ON THE BENEFITS OF A MONETARY UNION: DOES IT PAY TO BE BIGGER?∗

JOB MARKET PAPER

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October 2007

Abstract

A two area dynamic stochastic general equilibrium model is employed to investigate the welfare implications of losing monetary independence. Two policy regimes are compared: (i) in one area there is a common currency, while in the other area countries still retain their autonomous monetary policy; (ii) there are two monetary unions. When chosen by national authorities, monetary policy can stabilize optimally the effects of country-specific shocks. However, in that case, policy decisions internalize neither the spillover effects on the consumers living in the same area nor their impact on the world economy. Thus the adoption of a common currency implies a meaningful trade-off between the cost of not tailoring monetary policy to single country economic conditions and the gains entailed by the improvement upon the conduct of national monetary policies. Our results show that under markup shocks and plausible calibrations, there may be welfare gains from adopting a common currency.

Keywords: Optimal Monetary Policy, Currency Areas, Terms of Trade.
JEL Classification: E52, E61, F41.

∗I would like to thank Jordi Galí for his excellent supervision. For their helpful comments I thank Alessia Campolmi, Harald Fadinger, Stefano Gnocchi, Michael Reiter, Thijs Van Rens and all participants in seminars at Universitat Pompeu Fabra.
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1 Introduction

Which are the costs and the benefits of a monetary union? Should independent countries abandon their own currency to delegate monetary policy to a common central bank? These questions are far from new¹ but have been revitalized by the debate on the creation of the European Monetary Union (EMU). On theoretical grounds the costs of losing monetary autonomy are well known: in presence of nominal rigidities countries that share the same currency cannot properly stabilize asymmetric shocks. By contrast, the sources of welfare benefits that can rationalize the existence of a currency area have not been identified², at least if we restrict ourselves to the new open economy macroeconomic literature in which the objectives of the policymakers are fully microfounded³.

However there is a key aspect that until now seems to have been overlooked which can explain why a monetary union can be beneficial for its members. Especially, if we refer to the EMU experience, it is clear that the European Central Bank (ECB) sets the nominal interest rate for an economic area which is much bigger than each national economy. The difference in size can induce an improvement upon the conduct of the single country monetary policy in two respects. Firstly, being concerned about the welfare of all consumers living in the area, the common monetary authority internalizes the spillover effects that single country’s policymakers would generate inside the area if there were monetary autonomy. Secondly, being the union bigger, the policy decisions of its central bank have a much larger impact on the outside world. By realizing this, the larger central bank itself can better manage the policy externalities with regard to the rest of the world, especially with regard to other big economies like the United States.

The contribution of this paper is to verify whether, once these channels are taken into account, the adoption of a common currency can generate gains in terms of welfare that outweigh the costs of renouncing to monetary policy independence. To this end, I develop a dynamic stochastic general equilibrium open economy model in which the world is constituted by a continuum of small open regions as in Galí and Monacelli (2007). Each region produces a bundle of differentiated goods. Preferences exhibit home bias and the elasticity of substitution between home and foreign bundles is different from one. Prices are staggered implying a cost of the adoption of a common currency.

The regions are split in two areas, H and F. In area F all regions belong to a single country (as in the U.S.). Conversely area H is formed by a collection of sovereign small open economies (as in Europe). In this setup two different policy regimes (called A and B) are considered. Under regime A, in area H there are flexible exchange rates and each small open economy has its own autonomous central bank; under regime B in

¹See Mundell (1961).

²., at least from the macroeconomic point of view. As emphasized by the so called Delors report (1989), there are microeconomic benefits from adopting a common currency like for instance, saving in transaction costs. However it would be difficult to incorporate this kind of costs in a macroeconomics model.

³,...namely derived directly from the welfare of the representative household. See Rotemberg and Woodford (1997) and Benigno and Woodford (2005). There is a recent contribution of Corsetti (2007) on this issue who in a model with heterogeneous countries identifies the conditions under which monetary policy in a currency union is as efficient as under monetary autonomy.

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area $H$ there is a single currency and monetary policy is under the control of a common central bank (ECB). Instead in area $F$, independently of the policy regimes, all regions share a common currency and monetary policy is delegated to a single authority (FED). Moreover, in both regimes $A$ and $B$ monetary policies are optimal from the timeless perspective$^4$.

In this kind of setting, given the imperfect substitutability among goods produced in different countries, both small and big open economy policy authorities seek to exert their monopoly power on their terms of trade to other countries consumers’ expense$^5$. In particular, they would like to improve on average the terms of trade to render home produced goods more expensive$^6$. However, this incentive affects in a different way the optimal monetary policy decisions of the national authorities of the small open economies and of the central bank of the monetary union.

By taking as given what happens in the world economy, small open economy policymakers seek to improve their terms of trade (i.e. the average price of the goods produced in the world economy relatively to the average price of the goods produced within the region) by lowering average domestic output. This incentive explains why under the baseline calibration their monetary policy is more countercyclical in response to global markup shocks with respect to what would be efficient in a closed economy. In fact, by stabilizing more output than inflation, small open economy policymakers seek to decrease average (over time) domestic production. By contrast, the central bank of the monetary union internalizes the feedback effects of its policy decisions stemming from the global economy. As a result, it realizes that its terms of trade (i.e. the average price of the goods produced in the other area relatively to the average price of the goods produced within its own area) can be improved by lowering average domestic output or by increasing average foreign output. For this reason, under the baseline calibration it considers average foreign output too low and finds it optimal in response to global markup shocks to adopt a more procyclical policy allowing for an increase in the average (over time) foreign output.

These differences in incentives explain the differences in outcomes across policy regimes. In regime $B$ policymakers are exactly symmetric; thus under global markup shocks, they choose the same optimal monetary policy, thereby ensuring the same economic performance in the two areas. Conversely, in regime $A$, even when markup shocks are symmetric, there are relevant differences in optimal policy decisions. In response to negative markup shocks, small open economy policymakers adopt a more restrictive policy than the single central bank of the monetary union in order to stabilize domestic output. Therefore there is more deflation in area $H$ and output in area $F$ expands more than under regime $B$.

These differences across regimes explain why, despite the presence of idiosyncratic

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$^4$See Benigno and Woodford (2005) and Woodford (2003).

$^5$The effects of this externality are amplified by the hypothesis of home bias for both the bundles produced within the region and within the area as well as the assumption that the elasticity of substitution between home and foreign bundles is different from one. For a discussion see Pappa (2004), Benigno and Benigno (2003) and Obstfeld and Rogoff (2000). Notice that other policy instruments to affect terms of trade, such tariffs, cannot be used in the WTO.

$^6$The literature has already emphasized this incentive. See among others Corsetti and Pesenti (2001) and Epifani and Gancia (2005).
shocks, households of area $H$ can be better off by sharing a common currency. Indeed, I show that in the presence of markup shocks adopting the same currency may generate welfare benefits under reasonable calibrations. This finding is quite robust: even for relatively low level of the intertemporal elasticity of substitution between home and foreign bundles and high levels of the variance of the idiosyncratic shocks, welfare gains may be significant.

This paper is organized as follows. Section 2 describes the basic setup, section 3 determines the equilibrium conditions, section 4 formulates the optimal policy problems, section 5 describes the dynamic simulation and section 6 reports the results about the welfare evaluation.

## 2 The basic framework

The world consists of a continuum of small open regions indexed by $i \in [0, 1]^7$. The regions are subdivided in two areas, $H$ and $F$. In area $H$, there is a continuum of regions indexed by $i \in [0, \frac{1}{2})$, which are independent countries. Area $F$ consists of regions indexed by $i \in [\frac{1}{2}, 1]$, which belong to a single country. Each region produces a continuum of imperfect substitutable goods. Labour is immobile across both regions and areas.

### 2.1 Preferences

Agents are infinitely lived and maximize the expected value of the discounted sum of the period utility. Preferences of a generic region $i$ representative household are defined over a private consumption bundle, $C^i_t$ and labor $N^i_t(s)$:

$$\sum_{t=0}^{\infty} \beta^t E_0 \left[ \frac{C^i_{t+1} - N^i_t(s)^{\varphi+1}}{1 - \varphi} \right] \quad 0 < \beta < 1$$

(1)

where $\beta$ stands for the intertemporal preferences discount factor. Agents consume all the goods produced in the world economy. However, preferences exhibit home bias. The private consumption index is a CES aggregation of the following type:

$$C^i_t \equiv \left[ \frac{1}{\eta} C^i_{i,t} \left[ (\alpha_b - \alpha_s) \frac{\eta}{\sigma} C^i_{H,t} + (1 - \alpha_b) \frac{\eta}{\sigma} C^i_{F,t} \right] \right]^\frac{1}{\eta - 1} \quad i \in \left[ 0, \frac{1}{2} \right)$$

(2)

$$C^i_t \equiv \left[ \frac{1}{\eta} C^i_{i,t} \left[ (\alpha_b - \alpha_s) \frac{\eta}{\sigma} C^i_{H,t} + (1 - \alpha_b) \frac{\eta}{\sigma} C^i_{F,t} \right] \right]^\frac{1}{\eta - 1} \quad i \in \left[ \frac{1}{2}, 1 \right]$$

(3)

$\eta > 0$, $0 < \alpha_s < \alpha_b$ and $\frac{1}{2} < \alpha_b < 1$. $\alpha_s$ and $\alpha_b$ are the degrees of home bias for the goods produced within region $i$ and the area to which region $i$ belongs. Moreover, $\eta$ denotes the elasticity of substitution among $C^i_{H,t}$, $C^i_{F,t}$, $C^i_{i,t}$ which are defined as:

$$C^i_{H,t} \equiv \left[ \frac{1}{\eta} \int_0^{\frac{1}{2}} C^i_{j,t} \left[ j \frac{\eta}{\sigma} \right] dj \right]^\frac{1}{\eta - 1}$$

$$C^i_{F,t} \equiv \left[ \frac{1}{\eta} \int_{\frac{1}{2}}^{1} C^i_{j,t} \left[ j \frac{\eta}{\sigma} \right] dj \right]^\frac{1}{\eta - 1}$$

(4)

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$^7$This model is a general version of the basic framework layout by Galí and Monacelli (2005) and Galí and Monacelli (2007).
\[ C_{j,t}^i \equiv \left( \int_0^1 c_i^j(h^j) \frac{\alpha - 1}{\epsilon} dh^j \right)^\frac{\epsilon}{\alpha - 1} \quad j \in [0, \frac{1}{2}] \quad C_{j,t}^i \equiv \left( \int_0^1 c_i^j(f^j) \frac{\alpha - 1}{\epsilon} df^j \right)^\frac{\epsilon}{\alpha - 1} \quad i \in \left[ \frac{1}{2}, 1 \right] \] (5)

where \( \epsilon \) is the elasticity of substitution among goods produced in the same region. The definitions of the private consumption indexes (2), (4) and (5) enable us to determine consistent definitions of price indexes. In particular, \( P_{C^i,t} \), the consumers’ price index of region \( i \) is:

\[ P_{C^i,t} \equiv \left[ \alpha_s P_{t,t}^{1-\eta} + (\alpha_b - \alpha_s) P_{H,t}^{1-\eta} + (1 - \alpha_b) P_{F,t}^{1-\eta} \right]^\frac{1}{1-\eta} \quad i \in \left[ \frac{1}{2}, 1 \right] \] (6)

\[ P_{C^i,t} \equiv \left[ \alpha_s P_{t,t}^{1-\eta} + (\alpha_b - \alpha_s) P_{F,t}^{1-\eta} + (1 - \alpha_b) P_{H,t}^{1-\eta} \right]^\frac{1}{1-\eta} \quad i \in \left[ \frac{1}{2}, 1 \right] \] (7)

\[ P_{H,t} \equiv \left[ \frac{2}{\int_0^1 \alpha_t(v^t) \frac{\alpha - 1}{\epsilon} dv^t} \right]^{\frac{1}{\epsilon - 1}} \quad i \in \left[ 0, \frac{1}{2} \right] \quad P_{F,t} \equiv \left[ \frac{2}{\int_0^1 \alpha_t(v^t) \frac{\alpha - 1}{\epsilon} dv^t} \right]^{\frac{1}{\epsilon - 1}} \quad i \in \left[ \frac{1}{2}, 1 \right] \]

where all prices are denominated in the currency of the home country. Thus \( P_{t,t} \), \( P_{H,t} \) and \( P_{F,t} \) are producers’ price indexes. The law of one price is assumed to hold in all single good markets. However, given the home biased preferences, in general the purchasing power parity does not hold for indexes \( P_{C^i,t} \).

### 2.2 Consumption demand, portfolio choices and labor supply

The consumption and price index definitions allow to solve the consumer problem in two stages. In a first stage, agents decide how much real net income to allocate to buy goods produced at home and abroad. According to the set of optimality conditions, it is possible to determine agents’ demands as:

\[ C_{i,t}^i = \alpha_s \left( \frac{P_{t,t}}{P_{C^i,t}} \right)^{-\eta} C_{i,t}^i \quad C_{H,t}^i = (\alpha_b - \alpha_s) \left( \frac{P_{H,t}}{P_{C^i,t}} \right)^{-\eta} C_{i,t}^i \quad C_{F,t}^i = (1 - \alpha_b) \left( \frac{P_{F,t}}{P_{C^i,t}} \right)^{-\eta} C_{i,t}^i \quad i \in \left[ 0, \frac{1}{2} \right] \] (8)

\[ C_{i,t}^i = \alpha_s \left( \frac{P_{t,t}}{P_{C^i,t}} \right)^{-\eta} C_{i,t}^i \quad C_{F,t}^i = (\alpha_b - \alpha_s) \left( \frac{P_{F,t}}{P_{C^i,t}} \right)^{-\eta} C_{i,t}^i \quad C_{H,t}^i = (1 - \alpha_b) \left( \frac{P_{H,t}}{P_{C^i,t}} \right)^{-\eta} C_{i,t}^i \quad i \in \left[ \frac{1}{2}, 1 \right] \] (9)

and for \( i \in \left[ 0, \frac{1}{2} \right] : \)

\[ C_{j,t}^i = 2 \left( \frac{P_{j,t}}{P_{H,t}} \right)^{-\eta} C_{j,t}^i \quad j \in \left[ 0, \frac{1}{2} \right] \quad C_{j,t}^i = 2 \left( \frac{P_{j,t}}{P_{F,t}} \right)^{-\eta} C_{j,t}^i \quad j \in \left( \frac{1}{2}, 1 \right] \] (10)

\[ c_i^j(h^j) = \left( \frac{P_{i}^j}{P_{j,t}} \right)^{-\eta} C_{j,t}^i \quad j \in \left[ 0, \frac{1}{2} \right] \quad c_i^j(f^j) = \left( \frac{P_{i}^j}{P_{j,t}} \right)^{-\eta} C_{j,t}^i \quad j \in \left( \frac{1}{2}, 1 \right] \] (11)
while for $i \in \left( \frac{1}{2}, 1 \right]$:

$$
C^i_{t,t} = 2 \left( \frac{P_{j,t}}{P_{F,t}} \right)^{-\eta} C^i_{F,t} \quad j \in \left[ 0, \frac{1}{2} \right) \\
C^i_{j,t} = 2 \left( \frac{P_{j,t}}{P_{H,t}} \right)^{-\eta} C^i_{H,t} \quad j \in \left( \frac{1}{2}, 1 \right]
$$

(12)

$$
c^i_t(f^j) = \left( \frac{p_f(f^j)}{P_{j,t}} \right)^{-\varepsilon} C^i_{j,t} \quad j \in \left[ 0, \frac{1}{2} \right) \\
c^i_t(h^j) = \left( \frac{p_h(h^j)}{P_{j,t}} \right)^{-\varepsilon} C^i_{j,t} \quad j \in \left( \frac{1}{2}, 1 \right]
$$

(13)

The second stage coincides with the standard consumer problem. Agents maximize (1) with respect to $C^i_t$, $D^i_{t+1}$ and $N^i_t(s)$ subject to the following sequence of budget constraints:

$$
E_t[Q^i_{t,t+1}D^i_{t+1}] = D^i_t + W_{i,t}(s)N^i_t(s) - P_{C^i,t}C^i_t + T^i_t
$$

(14)

$$
N^i_t(s) = \left( \frac{W_{i,t}(s)}{W_{i,t}} \right)^{-\nu^i_t} N^i_t
$$

(15)

where:

$$
W_{i,t} = \left[ \int_0^1 W_{i,t}(s)^{1-\nu^i_t} ds \right]^{\frac{1}{1-\nu^i_t}}
$$

(16)

Condition (14) is the budget constraint which states that nominal saving, net of lump sum transfers has to equalize the nominal value of a state contingent portfolio. In fact, $W_{i,t}(s)$ stands for the per hour nominal wage, $Q^i_{t,t+1}$ denotes what is usually called the stochastic discount factor and $D^i_{t+1}$ is the payoff of one period maturity portfolio of firm shares.

Constraint (15) is a consequence of a CES aggregation of labor inputs which will be specified below and states that the labour market is monopolistic competitive. Indeed each agent offers a different kind of labour service. Thus $\nu^i_t$ stands for the elasticity of demand of labor which is time-varying and region-specific as in Clarida, Galí and Gertler (2002). Finally, (16) is simply the aggregate wage index. Domestic and international markets are assumed to be complete.

By the optimality conditions of the household problem:

$$
(1 + \mu^i_t)N^i_t(s)^{\sigma} C^i_t = \frac{W_{i,t}}{P_{C^i,t}}
$$

(17)

$$
\beta \left( \frac{C^i_{t+1}}{C^i_t} \right)^{-\sigma} \left( \frac{P_{C^i,t}}{P_{C^i,t+1}} \right) = Q^i_{t,t+1}
$$

(18)

which hold in all states of nature and at all periods and where $\mu^i_t \equiv \frac{1}{\nu^i_t-1}$. According to (17), workers set the real wage as markup over the marginal rate of substitution between consumption and leisure, while the value of the intertemporal marginal rate of substitution of consumption should equalize the stochastic discount factor expressed in terms of the currency of region $i$. Notice that since wages are perfectly flexible, $N^i_t(s) = N^i_t$ and $W_{i,t}(s) = W_{i,t}$ for all $s$ and $t$. 

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2.3 Firms, technology and price setting

In each region \( i \) there is a continuum of firms. Each of them produces a single differentiated good with a constant return to scale of the type:

\[
y_t(h^i) = A_i^t N_i(h^i)
\]

(19)

with \( N_i(h^i) = \left[ \int_0^1 N_i^i(s) \frac{v_i^i}{v_i^{i-1}} ds \right]^{\frac{\nu_i^i}{\nu_i^{i-1}}} \) being the labor input and \( A_i^t \) the region-specific technology shock. Given (19) and the fact that \( N_i^i = N_i^i(s) \) for all \( h^i \), the aggregate relationship between output and labor can be read as:

\[
N_i^i = \frac{Y_i^i}{A_i^t} Z_i^i
\]

(20)

where \( Y_i^i = \left[ \int_0^1 y_i^i(h^i) \frac{v_i^i}{v_i^{i-1}} dh^i \right]^{\frac{\nu_i^i}{\nu_i^{i-1}}} \) and \( Z_i^i = \int_0^1 \frac{v_i^i}{Y_i^i} dh^i \), and \( N_i^i = \int_0^1 N_i(h^i)dh^i \). Using (10) and (11) I will show below that \( Z_t^i = \int_0^1 \left( \frac{v_i^i(h^i)}{P_i^i} \right)^{\nu_i^i} dh^i \); thus \( Z_i^i \) can be interpreted as an index of the relative price dispersion across firms. We assume that good prices adjust according to a staggered mechanism à la Calvo. Therefore, in each period a given firm can reoptimize its price only with probability \( 1 - \theta \). As a result, the fraction of firms that set a new price is fixed and the aggregate producer price index of the intermediate goods evolves accordingly to:

\[
P^{i(1-\varepsilon)}_{i,t} = \theta P^{(1-\varepsilon)}_{i,t-1} + (1 - \theta) \tilde{p}_i(h^i)^{(1-\varepsilon)}
\]

(21)

with \( \tilde{p}_i(h^i) \) being the optimal price. Firms maximize the discounted expected sum of the future profits that would be collected if the optimal price could not be changed.

\[
\sum_{s=0}^{\infty} (\varepsilon)^s E_t \left\{ Q_{i,t+s} y_{i,t+s}(h^i) \left[ \tilde{p}_i(h^i) - MC_{i,t+s}^n \right] \right\}
\]

(22)

where \( y_i(h^i) = \left( \frac{P_i(h^i)}{P_i^{i-1}} \right)^{-\varepsilon} Y_i^i \) and \( MC_{i,t}^n = \left( \frac{1-\varepsilon}{A_i^t} \right)^{\nu_i^i} W_{i,t} \) is the nominal marginal cost with \( \tau^i \) denoting a constant labor subsidy. Taking into account (18) and that \( MC_{i,t} = \frac{MC_n}{P_{i,t}} \), the optimality condition of the firm problem can be written as:

\[
\sum_{s=0}^{\infty} (\beta \varepsilon)^s E_t \left\{ C_{i,t+s}^{\varepsilon} \left( \frac{\tilde{p}_i(h^i)}{P_{i,t+s}} \right)^{-\varepsilon} Y_{i,t+s} \frac{P_{i,t}}{P_{i,t+s}} \left[ \frac{\tilde{p}_i(h^i)}{P_{i,t}} - \varepsilon \frac{P_{i,t+s}}{P_{i,t}} MC_{i,t+s} \right] \right\} = 0
\]

(23)

Condition (23) states implicitly that firms reset their prices as a markup over a weighted average of the current and expected marginal costs, where the weight of the expected marginal cost at some date \( t + s \) depends on the probability that the price is still effective at that date.
3 Equilibrium

3.1 International risk sharing

The assumption of complete markets implies:

\[
\frac{C_i^{t-\sigma}}{P_{C,i,t}} = \frac{C_i^{t-\sigma}}{\varepsilon_{i,j,t} P_{C,j,t}}
\]  

(24)

for all \( t, i \in \left[0, \frac{1}{2}\right] \) and \( j \in \left(\frac{1}{2}, 1\right] \). According to (24), the value of marginal utility of consumption is equalized across regions. However, given the home bias in consumption, even if the law of one price holds, the purchasing power parity does not. As a consequence, consumption can be different across both regions and areas.

By properly integrating this equation we obtain:

\[
\frac{C_i^{t-\sigma}}{P_{C,i,t}} = \frac{C_{H,i,t}^{t-\sigma \min}}{\varepsilon_{i,j,t} P_{C,j,t}^\sigma} \quad i \in \left[0, \frac{1}{2}\right] \quad \frac{C_i^{t-\sigma}}{P_{C,i,t}} = \frac{C_{F,i,t}^{t-\sigma \min}}{\varepsilon_{i,F,t} P_{F,i,t}^\sigma} \quad i \in \left(\frac{1}{2}, 1\right]
\]  

(25)

for all \( i \), where \( \varepsilon_{i,j,t} \) stands for the nominal exchange rate of region \( j \) currency to region \( i \) currency\(^8\). Here \( C_{H,i,t}^{t-\sigma \min} \equiv \left[ 2 \int_0^1 C_i^{t-\sigma (1-\eta)} d\eta \right] \frac{1}{\sigma(1-\eta)} \), \( C_{F,i,t}^{t-\sigma \min} \equiv \left[ 2 \int_1^2 C_i^{t-\sigma (1-\eta)} d\eta \right] \frac{1}{\sigma(1-\eta)} \), \( P_{H,i,t}^\sigma \equiv \left[ 2 \int_0^1 \varepsilon_{H,j,t} P_{C,j,t}^{(1-\eta)} d\eta \right] \frac{1}{\sigma(1-\eta)} \) and \( P_{F,i,t}^\sigma \equiv \left[ 2 \int_1^2 P_{C,j,t}^{(1-\eta)} d\eta \right] \frac{1}{\sigma(1-\eta)} \).

Regarding conditions (25), notice the following. Within area \( F \), there is always a common currency, independently of the policy regime. Thus, \( \varepsilon_{F,i,t} = 1 \) for all \( i \in \left[\frac{1}{2}, 1\right] \). Conversely within area \( H \), \( \varepsilon_{H,i,t} = 1 \) for all \( i \in \left[0, \frac{1}{2}\right] \) only under regime B when there is a common currency and the exchange rates are fixed. Finally, in general, \( \varepsilon_{H,F,t} \) is floating under both regimes A and B. As shown in the appendix, it follows from to (25) and (24) that:

\[
\frac{P_{i,t}}{P_{C,i,t}} = \frac{\gamma_s + (\gamma_b - \gamma_s) \left( C_{H,i,t}^{t-\sigma (1-\eta)} \right)^{-\sigma (1-\eta)} + (1 - \gamma_b) \left( C_{F,i,t}^{t-\sigma (1-\eta)} \right)^{-\sigma (1-\eta)} \right] \frac{1}{\tau} \]  

(26)

for \( i \in \left[0, \frac{1}{2}\right] \) and where \( \gamma_s \equiv \frac{1}{\alpha_s} \) and \( \gamma_b \equiv \frac{-\alpha_s}{1 - 2\alpha_b} \). A corresponding condition can be retrieved for area \( F \):

\[
\frac{P_{i,t}}{P_{C,i,t}} = \frac{\gamma_s + (\gamma_b - \gamma_s) \left( C_{F,i,t}^{t-\sigma (1-\eta)} \right)^{-\sigma (1-\eta)} + (1 - \gamma_b) \left( C_{H,i,t}^{t-\sigma (1-\eta)} \right)^{-\sigma (1-\eta)} \right] \frac{1}{\tau} \]  

(27)

for all \( i \in \left(\frac{1}{2}, 1\right] \).

At the same time, (6) and (7) can be log-linearized as:

\[
\hat{p}_{i,t} - \hat{p}_{c,t} = - (\alpha_b - \alpha_s) \hat{s}_{t} H_{t} - (1 - \alpha_b) \hat{s}_{t} F_{t} \quad i \in \left[0, \frac{1}{2}\right]
\]  

(28)

\[
\hat{p}_{i,t} - \hat{p}_{c,t} = - (\alpha_b - \alpha_s) \hat{s}_{t} F_{t} - (1 - \alpha_b) \hat{s}_{t} H_{t} \quad i \in \left[\frac{1}{2}, 1\right]
\]  

(29)

\(^8\)... and \( \varepsilon_{H,j,t} \) stands for the nominal exchange rate of region \( j \) currency to a common unit of account of area \( H \).
where \( \hat{s}_{iH,t} \equiv c_{iH,t} + \hat{p}_{H,t} - \hat{p}_{i,t} \) and \( \hat{s}_{iF,t} \equiv c_{iF,t} + \hat{p}_{F,t} - \hat{p}_{i,t} \) denote the terms of trade of a small open economy \( i \) and areas \( H \) and \( F \) respectively\(^9\) and where \( \hat{c}_{iH,t} \equiv 2 \int_0^1 \hat{c}_i^1 dj \) and \( \hat{c}_{iF,t} \equiv 2 \int_1^2 \hat{c}_i^1 dj \). By combining (6) and (7) with (28) and (29) and using (25):

\[
\begin{align*}
\hat{s}_{iH,t} & = -\frac{\sigma}{\alpha_s} (\hat{c}_{iH,t} - \hat{c}_i^1) \quad i \in \left[0, \frac{1}{2}\right] \\
\hat{s}_{iF,t} & = -\frac{\sigma}{\alpha_s} (\hat{c}_{iF,t} - \hat{c}_i^1) \quad i \in \left[\frac{1}{2}, 1\right]
\end{align*}
\]

(30)

Moreover, by properly integrating the log-linear approximation of (26) and (28), it is easy to show that:

\[
\hat{s}_{HF,t} = -\sigma \left( \frac{1}{2\alpha_b - 1} \right) (\hat{c}_{F,t} - \hat{c}_{H,t})
\]

(31)

where \( \hat{s}_{HF,t} \equiv \hat{c}_{HF,t} + \hat{p}_{HF,t} - \hat{p}_{HF,t} \) stands for the terms of trade between area \( F \) and area \( H \). According to (31), in equilibrium a rise in the terms trade of the two areas reduces their relative consumption ratio as long as \( \alpha_b > 1 - \alpha_b \).\(^{11}\) A terms of trade worsening\(^{12}\) makes home consumers substitute the goods produced in area \( F \) with the goods produced in area \( H \) and increase their overall consumption because they relatively prefer the bundle produced in their own area. Notice that the impact of an improvement on the terms of trade on consumption differentials depends critically on the household relative risk aversion (or the inverse of the intertemporal elasticity of substitution of consumption \( \sigma \)). The higher is \( \sigma \), the lower is the difference in average consumption across areas associated with a movement in the terms of trade. More risk adverse households are more willing to share risk across different states of the world (or less willing to shift consumption across periods). Finally, by taking (30) in differences, it follows:

\[
\begin{align*}
\Delta c_{iH,t} + \pi_{H,t} - \pi_{i,t} & = -\sigma \gamma_s (\Delta \hat{c}_{H,t} - \Delta \hat{c}_i^1) \quad i \in \left[0, \frac{1}{2}\right] \\
\pi_{F,t} - \pi_{i,t} & = -\sigma \gamma_s (\Delta \hat{c}_{F,t} - \Delta \hat{c}_i^1) \quad i \in \left[\frac{1}{2}, 1\right]
\end{align*}
\]

(32) \hspace{1cm} (33)

Equation (33), and in regime \( B \) also equation (32), can be interpreted as a constraint imposed by the adoption of a common currency according to which, in response to asymmetric shocks, the terms of trade cannot adjust instantaneously because of the sluggish price adjustment and the fixed exchange rates. Conversely under regime \( A \) in area \( H \), when there is monetary autonomy, the fluctuations of the nominal exchange rates assure that condition (32) is always satisfied.

\(^9\)namely the average price of the goods produced in the small open economy \( i \) relative to the average price of the goods produced in areas \( H \) and \( F \). With a notational abuse \( \hat{p}_{i,t} \) indicates the log-deviation of the average price in area \( F \) expressed in terms of the common currency of that area. Similar interpretation applies to \( \hat{p}_{H,t} \).

\(^{10}\)We will use this as a general notation. For a given variable \( \hat{x}_t, \hat{x}_{H,t} \equiv 2 \int_0^{\frac{1}{2}} \hat{x}_i^1 dj \) and \( \hat{x}_{F,t} \equiv 2 \int_{\frac{1}{2}}^{1} \hat{x}_i^1 dj \).

\(^{11}\)That is \( \alpha_b > \frac{1}{2} \) as previously assumed.

\(^{12}\)namely an increase of \( \hat{s}_{HF,t} \).
3.2 IS curve

Given (18) and (25), we can recover the following condition for area \( F \):

\[
\frac{1}{1 + r_{F,t}} = \beta E_t \left\{ \left( \frac{C_{F,t+1}^i}{C_{F,t}^i} \right)^{-\sigma} \Pi_{F,t+1}^s \right\}
\]

(34)

where \( \frac{1}{1 + r_{F,t}} = E_t \{ Q_{t+1}^i \} \). When markets are complete, the expected value of the intertemporal marginal rate of substitution of private consumption, namely the price of a riskless portfolio, equalizes the price of the riskless bond, being \( r_{F,t} \) the nominal interest rate. The analogue of (34) for area \( H \) is

\[
\frac{1}{1 + r_{H,t}} = \beta E_t \left\{ \left( \frac{C_{H,t+1}^i}{C_{H,t}^i} \right)^{-\sigma} \Pi_{H,t+1}^s \right\}
\]

(35)

under regime \( A \) and:

\[
\frac{1}{1 + r_{H,t}} = \beta E_t \left\{ \left( \frac{C_{H,t+1}^i}{C_{H,t}^i} \right)^{-\sigma} \Pi_{H,t+1}^s \right\}
\]

(36)

otherwise. The log-linear approximation of conditions (34), (35) and (36) leads to:

\[
r_{F,t} - \rho = E_t \{ \pi_{F,t+1} \} - \sigma E_t \{ \Delta \hat{c}_{F,t+1} + (1 - \gamma_b)(\Delta \hat{c}_{H,t+1} - \Delta \hat{c}_{F,t+1}) \}
\]

(37)

\[
r_{H,t} - \rho = E_t \{ \pi_{H,t+1} \} - \sigma E_t \{ \Delta \hat{c}_{H,t+1} + (\gamma_b - \gamma_s)(\Delta \hat{c}_{H,t+1} - \Delta \hat{c}_{H,t+1}) + (1 - \gamma_b)(\Delta \hat{c}_{F,t+1} - \Delta \hat{c}_{H,t+1}) \}
\]

(38)

\[
r_{H,t} - \rho = E_t \{ \pi_{H,t+1} \} - \sigma E_t \{ \Delta \hat{c}_{H,t+1} + (1 - \gamma_b)(\Delta \hat{c}_{F,t+1} - \Delta \hat{c}_{H,t+1}) \}
\]

(39)

where \( \rho \equiv -\log(\beta) \). Conditions (37), (38) and (39) are the so called IS curves. Notice that under regime \( A \), \( r_{H}^i \) can be different across the regions in area \( H \) being national central banks independent in their policy decisions. Conversely under regime \( B \), \( r_{H}^i = r_{H,t} \) for all \( i \), being the nominal interest of area \( H \) set by the common central bank of the monetary union.

3.3 Aggregate demand

In each region \( i \) of area \( H \) the demand for a specific good, \( y_t(h^i) \), is determined by the demand of home and foreign consumers namely:

\[
y_t^i(h) = c_{t,t}^i(h) + \int_0^{1/2} c_{t,t}^i(h) dj + \int_{1/2}^1 c_{t,t}^i(h) dj
\]

(40)

for all \( i \in \left[ 0, \frac{1}{2} \right] \). Given (8), condition (40) can be read as:

\[
Y_t^i = \alpha_s \left( \frac{P_{t,t}}{P_{C^j,t}} \right)^{-\eta} C_t + 2(\alpha_b - \alpha_s) \int_0^{1/2} \left( \frac{P_{t,t}}{P_{C^j,t}} \right)^{-\eta} C_t^j dj + 2(1 - \alpha_b) \int_{1/2}^1 \left( \frac{P_{t,t}}{P_{C^j,t}} \right)^{-\eta} C_t^j dj
\]

(41)
with \( Y_t^i \equiv \left[ \int_0^1 y_t^i(h) \frac{dh}{h} \right]^{\frac{1}{\eta}} \). Because of (24), the aggregate demand for region \( i \) can be written as:

\[
Y_t^i = \left( \frac{P_{t,t}}{P_{C,t,t}} \right)^{-\eta} \left[ \alpha_s C_t^i + (\alpha_b - \alpha_s) C_t^{i,s} C_{H,t} + (1 - \alpha_b) C_t^{i,s} C_{F,t} \right]
\]  

(42)

with:

\[
C_{H,t} = 2 \int_0^{1/2} C_t^{1-\sigma_H} dj \quad C_{F,t} = 2 \int_{1/2}^1 C_t^{1-\sigma_F} dj
\]

(43)

for all \( i \in \left[ 0, \frac{1}{2} \right) \). A symmetric condition can be stated for all \( i \in \left( \frac{1}{2}, 1 \right] \), namely:

\[
Y_t^i = \left( \frac{P_{t,t}}{P_{C,t,t}} \right)^{-\eta} \left[ \alpha_s C_t^i + (\alpha_b - \alpha_s) C_t^{i,s} C_{H,t} + (1 - \alpha_b) C_t^{i,s} C_{F,t} \right]
\]

(44)

It is easy to show that the first order approximation of (42) and (44) corresponds to:

\[
\hat{y}_t^i = \hat{c}_t^i + (\delta_b - \delta_s)(\hat{c}_{H,t} - \hat{c}_t^i) + (1 - \delta_b)(\hat{c}_{F,t} - \hat{c}_t^i) \quad i \in \left[ 0, \frac{1}{2} \right)
\]

(45)

\[
\hat{y}_t^i = \hat{c}_t^i + (\delta_b - \delta_s)(\hat{c}_{H,t} - \hat{c}_t^i) + (1 - \delta_b)(\hat{c}_{F,t} - \hat{c}_t^i) \quad i \in \left[ \frac{1}{2}, 1 \right]
\]

(46)

where \( \delta_s \equiv \gamma_s \eta^\sigma + \alpha_s (1 - \eta^\sigma) \) and \( \delta_b \equiv \gamma_b \eta^\sigma + \alpha_b (1 - \eta^\sigma) \). According to (45), the aggregate demand of goods produced in region \( i \) depends directly on the terms of trade (through (30)). Any terms of trade improvement\(^{13}\) between region \( i \) and areas \( H \) or \( F \) switches the expenditure of both home and foreign households toward foreign goods. Aggregating (45) and (46), we obtain:

\[
\hat{y}_{H,t}^i = \hat{c}_{H,t} + (1 - \delta_b)(\hat{c}_{F,t} - \hat{c}_{H,t}) \quad i \in \left[ 0, \frac{1}{2} \right)
\]

(47)

\[
\hat{y}_{F,t}^i = \hat{c}_{F,t} + (1 - \delta_b)(\hat{c}_{H,t} - \hat{c}_{F,t}) \quad i \in \left[ \frac{1}{2}, 1 \right]
\]

(48)

### 3.4 Aggregate supply

Given condition (23), the optimal price is determined as:

\[
\tilde{p}_t^i(h_t) = K_t^i \frac{P_{t,t}}{F_t^i}
\]

(49)

with:

\[
K_t^i \equiv \sum_{s=0}^{\infty} (\beta \theta)^s E_t \left[ C_t^{i,t+s} - \sigma Y_t^{i,t+s} \left( \frac{P_{t,t+s}}{P_{C,t,t+s}} \right)^{\frac{\varepsilon - 1}{\varepsilon - 1}} M C_{t,t+s} \right]
\]

(50)

\[
F_t^i \equiv \sum_{s=0}^{\infty} (\beta \theta)^s E_t \left[ C_t^{i,t+s} - \sigma Y_t^{i,t+s} \left( \frac{P_{t,t+s}}{P_t^i} \right)^{\frac{\varepsilon - 1}{\varepsilon - 1}} \right]
\]

(51)

\(^{13}\)namely a decrease of \( \hat{s}_{H,t} \) or \( \hat{s}_{F,t} \).
which can be read as:

\[ K^i_t = C_i^{i-\sigma}Y_i^i \frac{P_{i,t}}{P_{C_i,t}} \varepsilon - 1 MC_{i,t} + \beta \theta E_t \{ \Pi_{i,t+1}^\varepsilon K_{i+1}^i \} \] (52)

\[ F_t^i = C_i^{i-\sigma}Y_i^i \frac{P_{i,t}}{P_{C_i,t}} + \beta \theta E_t \{ \Pi_{i,t+1}^{\varepsilon-1} F_{i+1}^i \} \] (53)

where \( \Pi_{i,t} \equiv \frac{P_{i,t}}{P_{i,t-1}} \). Following Benigno and Woodford (2005), from (49) and (21) we can retrieve the next conditions:

\[ \frac{1 - \theta \Pi_{i,t-1}^{\varepsilon-1}}{1 - \theta} = \left( \frac{F_t^i}{K_t^i} \right)^{\varepsilon-1} \] (54)

\[ Z_t^i = \theta Z_{t-1}^i \Pi_{i,t}^\varepsilon + (1 - \theta) \left( \frac{1 - \theta \Pi_{i,t-1}^{\varepsilon-1}}{1 - \theta} \right)^{\varepsilon-1} \] (55)

By the log-linear approximation of (17) (50) (51) and (55):

\[ \pi_{i,t} = \lambda \hat{mc}_t^i + \beta E_t \{ \pi_{i,t+1} \} \] (56)

with \( \lambda \equiv \frac{(1-\theta)(1-\theta)}{\theta} \) and where:

\[ \hat{mc}_t^i = (\hat{w}_t^i - \hat{p}_t^i) - (\hat{p}_t^i - \hat{p}_t^{i,c,t}) - \hat{\sigma}_t^i \]

for all \( t \) and \( i \). Condition (56) is the New Keynesian Phillips Curve which results from the Calvo mechanism. As usual, current domestic inflation depends on the expectation on future domestic inflation and the current real marginal cost of producing goods. Being the economy open in equilibrium this cost is determined by the real wage, which is equal to the marginal rate of substitution between consumption and leisure, the labour productivity and the product price index relative to the consumption price index (26) and (27). By substituting (28) and log-linear approximation of (26) we obtain:

\[ \hat{mc}_t^i = \varphi \hat{y}_t^i + \sigma c_t^i + (\alpha_b - \alpha_s) \hat{s}_{i,H,t} + (1 - \alpha_b) \hat{s}_{i,F,t} - (1 + \varphi) \hat{a}_t^i + \mu_t^i \]

\[ = \varphi \hat{y}_t^i + \sigma c_t^i + \sigma \left[ (\gamma_b - \gamma_s) \hat{c}_{i,H,t} - \hat{c}_t^i \right] + (1 - \gamma_b) \left( \hat{c}_{i,F,t} - \hat{c}_t^i \right) \]

\[ + (1 + \varphi) \hat{a}_t^i + \mu_t^i \] (58)

for all \( i \in \left[ 0, \frac{1}{2} \right] \). According to (58), an improvement of the terms of trade of region \( i \) lowers firms’ real marginal costs. Given (58), we can rewrite condition (56) for \( i \in \left[ 0, \frac{1}{2} \right] \) and its symmetric condition for \( i \in \left[ \frac{1}{2}, 1 \right] \) as:

\[ \pi_{i,t} = \lambda \left[ \varphi \hat{y}_t^i + \sigma \left( \gamma_s c_t^i + (\gamma_b - \gamma_s) \hat{c}_{i,H,t} + (1 - \gamma_b) \hat{c}_{i,F,t} \right) - (1 + \varphi) \hat{a}_t^i + \mu_t^i \right] + \beta E_t \{ \pi_{i,t+1} \} \] (59)

\[ \pi_{i,t} = \lambda \left[ \varphi \hat{y}_t^i + \sigma \left( \gamma_s c_t^i + (\gamma_b - \gamma_s) \hat{c}_{i,F,t} + (1 - \gamma_b) \hat{c}_{i,H,t} \right) - (1 + \varphi) \hat{a}_t^i + \mu_t^i \right] + \beta E_t \{ \pi_{i,t+1} \} \] (60)

Under regime A the rational expectation stochastic equilibrium is characterized by (38), (45) and (59) for all \( i \in \left[ 0, \frac{1}{2} \right] \) and by (33), (37), (46) and (60) for all \( i \in \left[ \frac{1}{2}, 1 \right] \), while under regime B by (32), (39), (45) and (59) for all \( i \in \left[ 0, \frac{1}{2} \right] \) and by (33), (37), (46) and (60) for all \( i \in \left[ \frac{1}{2}, 1 \right] \).

It remains to determine the optimal monetary policy.

\[ \text{namely a decrease of } \delta_{i,H,t} \text{ or } \delta_{i,F,t}. \]
4 Optimal monetary policy problems

As anticipated in the introduction, in order to evaluate the costs and the benefits of a monetary union, we can compare two policy regimes. Under regime $A$, while there is a monetary union in area $F$, area $H$ is populated by a continuum of countries with different currencies; under regime $B$, there is a monetary union in both areas $F$ and $H$.

The optimal policy problems under the two regimes are solved by using the linear quadratic approach pioneered by Benigno and Woodford (2005), which allows to find the timelessly optimal policies. Following this approach, we determine first the zero inflation deterministic steady state. Then, we retrieve the quadratic welfare approximation for the monetary authorities of both the small open economy and the monetary unions, employing the second order approximation of the structural equations. Finally, we find the optimal policies by maximizing these quadratic approximations subject to the equilibrium conditions approximated to the first order.

4.1 The deterministic steady state

The steady state level of output is determined by a constant and generic labour subsidy \( \tilde{\tau} = 1 - \frac{1 - \tau}{1 + \mu \sigma} \). We assume that \( \tilde{\tau} \) is equal across countries. As shown in the appendix, under this assumption for any \( \tilde{\tau} \) there exists a symmetric deterministic steady state at which zero inflation is a Nash equilibrium policy\(^{15} \) for all policymakers in areas $H$ and $F$ under both regimes $A$ and $B$. Thus, the equilibrium equations and the objectives of the policymakers are approximated around the following steady state:

\[
Y = (1 - \tilde{\tau})^{-\frac{1}{\sigma + \tau}}
\]  

(61)

where the level \( \tilde{\tau} \) is not specified ex ante\(^{16} \).

Still, there are two special cases of interest. In particular, in this section we determine the desired levels of steady state output of both the small open economy authorities and the central banks of the monetary union. Given their different incentives, these policymakers consider efficient two different steady states. As a consequence, they have different "perceptions" of the steady state distortion. And, as it will be made clear

---

\(^{15}\)As it is clarified in the appendix, the non-linear optimal policy problems are formulated as follows. The policymaker of the small open economy $i$ maximizes the welfare of her households, with respect to $C_i^t$, $Y_i^t$, $Z_i^t$, $K_i^t$, $F_i^t$ and $\Pi_{i,t}$ subject to (42), (51), (50), (55) and where $P_{i,t}/P_{C_i,t}$ are determined consistently with (26), while $C_H^t$, $C_F^t$, $C_{H,t}$ and $C_{F,t}$ are taken as given. Conversely, the policymaker of the monetary union maximizes the average union welfare with respect to $C_i^t$, $Y_i^t$, $Z_i^t$, $K_i^t$, $F_i^t$, for all $i$ and with respect to $\Pi_{i,t}$ for all regions $i$ belonging to the monetary union. The constraints of policy problem are equations (42),(44), (51), (50), (55) for all $i$ and the non-linear version of (33) (and in regime $B$ even the non-linear version of (32)), where $P_{i,t}/P_{C_i,t}$ are determined consistently with (26) and (27) while the state contingent path(s) of inflation chosen by the other policymaker(s) is (are) taken as given. The solutions to these policy problems allow to determine the Nash equilibrium policies. The assumption that policymakers’ strategies are specified in terms of inflation rates is conformed to Benigno and Benigno (2003) and Benigno and Benigno (2006). This hypothesis is consistent even with the optimal policy problem of the small open economy policymakers. Indeed, if the economy is small, the average union variables can be determined thanks to the monetary policy chosen by the other policymakers jointly of the equilibrium equations.

\(^{16}\)and at which $A = 1$. 

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in the next sections, this same divergence explains some of the differences in optimal policy decisions over the business cycle. In the case of the small open economy \( i \), the desired steady state output can be retrieved by maximizing:

\[
\sum_{t=0}^{\infty} \beta^t E_0 \left[ \frac{C_i^{1-\sigma}}{1-\sigma} - \frac{1}{\varphi + 1} \left( \frac{Y_i^i}{A_i^i} \right)^{\varphi+1} \right]
\]

with respect to \( C_i^i \) and \( Y_i^i \), subject to (42) where \( P_{i,t}/P_{C_i,t} \) are determined consistently with (26), while \( C_{H,t}, C_{F,t}^*, C_{H,t} \) and \( C_{F,t} \) are taken as given. According to the first order conditions at the symmetric deterministic steady state:

\[
Y_s = \delta_s \frac{1}{\sigma + \varphi}
\]

(62)

where, as above, \( \delta_s \equiv \gamma_s \eta \sigma + \alpha_s (1 - \eta \sigma) \) which is always greater than one. The optimal labour subsidy that allows to implement this allocation is given by:

\[
\tilde{\tau}_s = 1 - \delta_s
\]

(63)

Thus, the small open policymakers would not like to reach the Pareto efficient steady state at which the monopolistic distortions are exactly eliminated\(^{17}\). They would rather prefer a lower level of steady state production. Indeed, given the imperfect substitutability between home and foreign bundles, they recognize they can improve their terms of trade, by reducing the production of domestic goods\(^{18}\). In other words, they try to increase the utility of domestic households by rendering home produced goods more expensive and externalizing in this way labour effort to other countries’ workers.

Notice that \( \delta_s \) depends positively on \( \eta \), the elasticity of substitution between home and foreign goods, \( \sigma \), the relative risk aversion coefficient and \( 1 - \alpha_s \) the degree of openness of the small country. The larger are these parameters, the stronger is the externality generated by the small open economy policymakers.

In the case of the policymaker of the monetary union, the desired steady state output can be determined by maximizing:

\[
\sum_{t=0}^{\infty} \beta^t E_0 \left[ \int_{0}^{\tilde{\tau}} \left( \frac{C_i^{1-\sigma}}{1-\sigma} - \frac{1}{\varphi + 1} \left( \frac{Y_i^i}{A_i^i} \right)^{\varphi+1} \right) \mathrm{d} \tilde{\tau} \right]
\]

(64)

with respect to \( C_i^i \) and \( Y_i^i \) for all \( i \in [0, 1] \) subject to:

\[
\frac{P_{i,t}}{P_{C_i,t}} = \left( 1 - \tilde{\tau} \right) \frac{Y_i^i}{A_i^{1+\varphi}} C_i^{\sigma}
\]

(65)

for all \( i \in [\frac{1}{2}, 1] \), (42) and (44) and where \( P_{i,t}/P_{C_i,t} \), \( C_{H,t} \) and \( C_{F,t} \) are determined according to (26), (27) and (43)\(^{19}\). From the first order conditions of this problem it

\(^{17}\)The Pareto efficient allocation corresponds to \( Y = 1 \) which can be achieved by setting \( \tilde{\tau} = 0 \).

\(^{18}\)And in fact the optimal labour subsidy is set equal to \( 1 - \delta_s \), a parameter related with the average elasticity of the domestic goods demand with respect to the terms of trade of the small open economy. As made clear by (45) and (59), two are the relevant terms of trade from the small open economy point of view: those of the small open economy and areas \( H \) and \( F \).

\(^{19}\)Implicitly (65) states that the policymaker of area \( H \) takes as given the strategy \( \tilde{\tau} \) chosen by a symmetric policymaker in area \( F \).
follows that at the symmetric steady state:

$$Y_b = \left[ 1 - \frac{(1 - \delta_b)(\sigma + \varphi)}{(\delta_b \varphi + \gamma_b \sigma)} \right]^{\frac{1}{(\sigma + \varphi)}}$$  \hspace{1cm} (66)

This allocation can be achieved by setting the labour subsidy:

$$\tilde{\tau}_b = \frac{(1 - \delta_b)(\sigma + \varphi)}{(\delta_b \varphi + \gamma_b \sigma)}$$  \hspace{1cm} (67)

where $\delta_b \equiv \gamma_b \eta \sigma + \alpha_b (1 - \eta \sigma)$ which is always greater than 1. According to these conditions, even in the case of the big economy, the policymaker tries to influence the terms of trade in her favour by reducing her production with respect to what would be Pareto efficient. However by the comparing (66) and (62), it can be shown that\textsuperscript{20}:

$$1 > Y_b > Y_s$$  \hspace{1cm} (68)

Thus, in the symmetric steady state, the policymaker of the monetary would choose a level of output higher than that considered efficient by the small open economy authorities. The reasons for this are twofold. First of all, bigger countries are less open. This lowers the incentive of their policymakers to improve the terms of trade. Secondly, big economy authorities try to influence only the terms of trade across areas\textsuperscript{21}. Therefore, on the one hand they internalize the external effects produced within the monetary union. On the other hand, they take into account the impact that their actions produce on the foreign economy. As a consequence, they are aware, for instance, that an improvement in their terms of trade increases the demand of foreign goods. All these effects go in the direction of reducing the real distortions generated by the desire of influencing their terms of trade itself.

Summing up, the difference in size between small and big countries affects the incentives of their policymakers and thus the desired level of steady state output. As explained in the next sections, this also has an impact on their optimal policies over the business cycle.

### 4.2 The case of the small open economy policymaker

As shown in the appendix, the objective of the small open economy policymaker of country $i$ in area $H$ can be approximated up to the second order as:

$$-\frac{1}{2} Y^\psi + \sum_{t=0}^{\infty} \beta^t E_0 \left[ (1 - \zeta_s(\varphi + 1)) \frac{\varepsilon}{\lambda} (\pi_{i,t})^2 + (1 - \zeta_s(\varphi + 1)) \varphi (\tilde{g}_{i,s}^{ resisted})^2 - 2 \zeta_s \gamma_s \sigma (\tilde{e}_{i,s}^{2,\delta})^2 + (1 - \varpi)(\sigma - 1) + \zeta_s \gamma_s \sigma^2 + (1 - \zeta_s \varphi) (\delta_s + \omega_1) (\tilde{e}_{i,s}^{2,\delta})^2 \right] + t.i.p. \hspace{1cm} (69)$$

where $\zeta_s \equiv \delta_s - (1 - \varpi) \frac{\delta_s \varphi + \gamma_s \sigma}{\delta_s \varphi + \gamma_s \sigma}$, $\omega_1$ is properly defined in the appendix and $\tilde{x}_{i,i}^{2,\delta} \equiv \tilde{x}_{i} - \tilde{x}_{i}^{ resisted}$. \tilde{x}_{s,t} indicates the target of the small open economy monetary authority which is determined

\textsuperscript{20}At least under the baseline calibration.

\textsuperscript{21}This explains the dependence of $\tilde{\tau}_b$ on $1 - \delta_b$, a parameter that governs the elasticity of aggregate demand for the domestic goods to the terms of trade across areas.
by (112) - (115). Moreover \(i.p\) stands for terms independent of monetary policy that include the aggregate variables of both the areas \(H\) and \(F\).

The welfare approximation in (69) contains the output gap, the consumption gap and inflation like in a closed economy. What is crucially different are the weights attached to these variables and the target that the authority would like to implement. This divergence with respect to the closed economy case is explained again by the desire of open economy policymakers to influence their terms of trade in their favour. In fact, on the one hand, this incentive works even over the business cycle and gives reason, for instance, for the higher weight attributed to consumption gap volatility: policymakers realize that fluctuations in consumption are associated with fluctuations in the terms of trade. On the other hand, this same incentive explains why from the small open economy policymakers viewpoint, the steady state is efficient as long as \(Y = Y_s\). Indeed, as discussed above, at the steady state open economy policymakers prefer a low level of steady state output to render home produced goods more expensive. This has clear consequences for the weights in (69), given that \(\zeta_s\) depends critically on the difference between \(\hat{\tau}\) and \(\hat{\tau}_s\).

In order to better understand in which direction the openness of the small economy modifies the conduct of monetary policymakers, it is useful to consider the following “targeting” condition:

\[
(1 - \zeta_s(\varphi + 1))\bar{mc}_{H,t} = \zeta_s(\varphi + 1)\bar{\mu}_{H,t} + \kappa_s\hat{s}_{HF,t}
\]  

(70)

where \(\kappa_s\) is a proper combination of the deep parameters of the model and \(\bar{mc}_{H,t}\) is the average firms marginal cost in area \(H\) in the absence of markup shocks, namely \(\bar{mc}_s \equiv \varphi\hat{\mu}_{H,t} + \sigma\hat{c}_{H,t} - (\varphi + 1)\hat{\alpha}_{H,t} + \sigma(1 - \gamma_h)(\hat{c}_{F,t} - \hat{c}_{H,t})\). Condition (70) is retrieved by properly rearranging and integrating the equations determining the target (112) - (115). It clarifies when monetary authorities desire to stabilize the average marginal rate of substitution between consumption and leisure at the level that would be efficient in a first best environment (i.e. \(\bar{mc}^*_{H,t} = 0\)). In fact, according to (70), if area \(H\) is closed with respect to the rest of the world (i.e. \(\alpha_h = 1\) which implies \(\kappa_s = 0\)) and there are no markup shocks, the real marginal cost, \(\bar{mc}_{H,t}^*\) and the inflation are on average completely stabilized under the optimal monetary policies. However, in general, as long as area \(H\) is open, this result is not achieved for two reasons.

Firstly, as indicated by the terms \(\kappa_s\hat{s}_{HF,t}\), even in the absence of markup shocks, in general, the incentive to affect the terms of trade conditions the optimal monetary policy decisions even over the business cycle.

Secondly and most importantly, the term \(\zeta_s(\varphi + 1)\bar{\mu}_{H,t}\) says that in the presence of domestic markup shocks, even the steady state distortion from the small open economy point of view induces monetary policymakers to not seek to completely stabilize \(\bar{mc}_{H,t}\). Indeed, only if the steady state is considered efficient by the monetary authorities (i.e.

\[\hat{\tau} = \hat{\tau}_s\]

\[\zeta_s = 0\]

\[2\alpha_h - 1\]

\[\delta_s^{-1}\]

\[\sigma(1 - \gamma_h)\delta_h - \omega_2 + \zeta_s\gamma_h((1 - \delta_h) - \sigma(1 - \gamma_h))]\]

\[\sigma(1 - \gamma_h)\delta_h - \omega_2 + \zeta_s\gamma_h((1 - \delta_h) - \sigma(1 - \gamma_h))]\]

\[\sigma(1 - \gamma_h)\delta_h - \omega_2 + \zeta_s\gamma_h((1 - \delta_h) - \sigma(1 - \gamma_h))]\]

\[\sigma(1 - \gamma_h)\delta_h - \omega_2 + \zeta_s\gamma_h((1 - \delta_h) - \sigma(1 - \gamma_h))]\]

\[\sigma(1 - \gamma_h)\delta_h - \omega_2 + \zeta_s\gamma_h((1 - \delta_h) - \sigma(1 - \gamma_h))]\]
$$\hat{\tau} = 1 - \delta_s$$, namely $$\varsigma_s = 0$$, then their target is not affected by domestic markup shocks. At that steady state any shock of this kind brings about an inefficient wedge between the real wage and the marginal rate of substitution between consumption and labour. As a result, the independent central banks would like to completely stabilize output and all real variables\(^{27}\). Nevertheless, when $$Y \neq Y_s$$, this is not true any more. The target of the small open economy policymaker reacts to domestic markup shocks. In fact, if as under the baseline calibration the steady state output is inefficient high (i.e. $$\hat{\tau} > 1 - \delta_s$$ and $$\varsigma_s > 0$$), then at the margin an increase of leisure rises utility by more than an increase of consumption. This generates a motive for the monetary authorities to seek to influence the average per-period output\(^{28}\) and to change their inflation output trade-off. In fact, by being more countercyclical\(^{29}\) in response to domestic markup shocks, the monetary policymakers of the small open economies can decrease the average per-period output which is too high from their perspective\(^{30}\). As a result, under the baseline calibration they tend to stabilize more output than inflation in the case in which the steady state is efficient in their perspective.

The timelessly optimal monetary policy can be retrieved by maximizing (69) with respect to $$\tilde{y}^{i,s}_t$$, $$\tilde{c}^{i,s}_t$$ and $$\pi_{i,t}$$ subject to the following sequence of constraints:

$$\tilde{y}^{i,s}_t = \delta_s \tilde{c}^{i,s}_t$$ (71)

$$\pi_{i,t} = \lambda \left( \varphi \tilde{y}^{i,s}_t + \gamma \tilde{c}^{i,s}_t \right) + \tilde{v}^{i,s}_t + \beta E_t \{ \pi_{i,t+1} \}$$ (72)

for all $$t$$ where $$\tilde{v}^{i,s}_t = \lambda \left( \varphi \tilde{y}^{i,s}_t + \gamma \tilde{c}^{i,s}_t + \sigma \gamma_b \gamma_s \tilde{c}^{i,s}_t + \sigma (1 - \gamma_b) \tilde{c}_{H,t} + \sigma (1 - \gamma_b) \tilde{c}_{F,t} - (1 + \varphi) \tilde{a}_t + \tilde{\mu}_t \right)$$.

### 4.3 The case of the policymaker of the monetary union

As shown in the appendix, if there is a monetary union the in area $$H$$, the objective of the monetary policymaker can be approximated in a purely quadratic way as:

$$-\frac{1}{2} Y^{\varphi + 1} \sum_{t=0}^{\infty} \beta^t E_0 \left[ (1 - \varsigma_b (\varphi + 1)) \frac{\pi_{H,t}}{\lambda} \right] + (1 - \varsigma_b (\varphi + 1)) \varphi \tilde{y}^{i,s}_{H,t} \right)^2$$

$$\varsigma_b \sigma \left( \gamma_b \tilde{c}^{i,s}_{H,t} + (1 - \gamma_b) \tilde{c}^{i,s}_{F,t} \right) \tilde{y}^{i,s}_{H,t} + \phi H (\tilde{c}^{i,s}_{H,t})^2 - (\xi - \varsigma_b (\varphi + 1)) \varphi \tilde{y}^{i,s}_{H,t} \right)^2$$

$$- (\xi - \varsigma_b) \sigma \left( \gamma_b \tilde{c}^{i,s}_{F,t} + (1 - \gamma_b) \tilde{c}^{i,s}_{H,t} \right) \tilde{y}^{i,s}_{F,t} + \phi F (\tilde{c}^{i,s}_{F,t})^2 + \phi H \tilde{c}^{i,s}_{H,t} \tilde{c}^{i,s}_{F,t} \right] + t.i.p. (73)$$

\(^{27}\) However, given the staggered price adjustment, this would never be possible.

\(^{28}\) Indeed, because of the steady state distortion, the second order approximation of the utility of the households depends on the expected value of output fluctuations. At the same time, this expected value is negatively affected by the covariance between output fluctuations and markup shocks (as emphasized by the second order approximation of the supply curve recovered in the appendix). As a result, the policymaker recognizes that by enhancing the negative (positive) correlation between markup shock and output, she can increase (decrease) the expected value of output fluctuations.

\(^{29}\) At least as long as \(1 - \varsigma_b (\varphi + 1) > 0\).

\(^{30}\) Clearly if the steady state output were inefficiently low (i.e. \(\hat{\tau} < 1 - \delta_s\) and \(\varsigma_s < 0\)), then the reverse reasoning applies: at the margin it is more beneficial to increase consumption than leisure. And in this case the small open economy policymakers would like to increase the average per-period output.
where \( \zeta_b \equiv \frac{1}{2} \frac{\bar{\tau}}{\sigma + \varphi} - \frac{\delta_b - 1 + (1/2) \bar{\tau}}{\sigma + (1 - 2\gamma_b) \varphi} \), \( \xi \equiv \frac{\bar{\tau}}{\sigma + \varphi} - \frac{\delta_b - 1 + (1/2) \bar{\tau}}{\sigma + (1 - 2\gamma_b) \varphi} \). \( \xi \) and \( \ddot{x}_t \equiv \dot{x}_t - \dot{x}_b^b \). \( \hat{x}_b^b \) denotes the target of the monetary union central bank which can be determined from (47)-(48) and (131)-(135)\(^{31}\). These last conditions are derived in the appendix where even \( \varrho_H \), \( \varrho_F \) and \( \varrho_{H,F} \) are properly defined. In addition, \( t.i.p \) stands for terms independent of policy and includes the state contingent path of \( \pi_{F,t} \) decided by the policymaker of the monetary union in area \( F \) and the differentials between country specific and average union variables\(^{32}\). By comparing (69) and (73), two main differences emerge both due to the larger size of the monetary union with respect to the small open countries. Indeed, on the one hand, since the monetary union is bigger, its central bank internalizes the feedback effects of its policy decisions stemming from the foreign economy. This explains the dependence of the welfare approximation in (73) on foreign output and consumption gaps. On the other hand, some of the weights of the variables entering in that approximation are governed by the difference between \( \bar{\tau}_b \) and \( \bar{\tau}^33\). Indeed, from the point of view of the authority of the monetary union the steady state is at the desired level as long as \( Y = Y_b \).

The inspection of the following ”targeting” condition:

\[
(1 - \zeta_b (\varphi + 1)) \overline{m^H_{H,t}} - (\xi - \zeta_b) (\varphi + 1) \overline{m^b_{F,t}} = \zeta_b (\varphi + 1) \bar{\mu}_{H,t} + (\xi - \zeta_b) (\varphi + 1) \bar{\mu}_{F,t} + \kappa_b \ddot{\sigma}_{H,F,t}
\]

(74)
gives some insights about the incentive of monetary union’s policymakers. \( \overline{m^H_{H,t}} \) and \( \overline{m^b_{F,t}} \) denote the average area firms marginal costs in the absence of markup shocks\(^{34}\), while \( \kappa_b \)\(^{35}\) is a proper combination of the deep parameters of the model. Like its analogue (70), condition (74) is derived from the equations that determine the target of the monetary union policymakers, namely (47)-(48) and (131)-(135).

Condition (74) can be interpreted as follows. As in the case of the small open economy, the common central bank in area \( H \) balances the need to stabilize the average real marginal cost in its area with a twofold desire: on the one hand, it wants to condition the terms of trade in its favor; on the other hand, in the presence of domestic markup shocks, it seeks to influence the average per-period domestic output depending on the steady state distortion\(^{36}\).

However, according to (74), the target of the policymaker of the monetary union differs from that of the small open economy authority in an important respect. By realizing how its decisions affect the demand and the supply of foreign goods, the monetary authority of the currency area cares even about the stabilization of the

\(^{31}\)namely the constraint efficient allocation from the perspective of the policymaker of area \( H \). This allocation corresponds to the allocation chosen by a planner that maximizes (73) subject exclusively to the resource constraints (47) and (48). See the appendix.

\(^{32}\)Indeed, by choosing the average union inflation, the common central bank can influence only the average union performance. However, these terms have to be taken into account for the welfare evaluation.

\(^{33}\)Indeed, when \( \bar{\tau} = \bar{\tau}_b \) then \( \zeta_b = 0 \).

\(^{34}\)that is \( \overline{m^H_{H,t}} = \varphi \bar{y}^H_{H,t} + \sigma \bar{c}^H_{H,t} - (\varphi + 1) \bar{a}_{H,F} + \sigma (1 - \gamma_b) \bar{c}^L_{H,t} - \bar{c}^b_{H,t} \) and \( \overline{m^b_{F,t}} = \varphi \bar{y}^b_{F,t} + \sigma \bar{c}^b_{F,t} - (\varphi + 1) \bar{a}_{F,F} + \sigma (1 - \gamma_b) \bar{c}^F_{F,t} - \bar{c}^b_{F,t} \).

\(^{35}\)namely \( \kappa_b \equiv (2 \alpha_b - 1) \sigma^{-1} (1 - \zeta_b (\varphi + 1) (1 - \delta_b) - \alpha (1 - \gamma_b)) - (\xi - \zeta_b) (\varphi + 1) (1 - \gamma_b) \).

\(^{36}\)Under the baseline calibration the steady state domestic output is inefficiently low from its perspective. This generates a motive to adopt a more procyclical policy with respect to what would be optimal at the steady state efficient from its point of view. See the above discussion.
average firms’ real marginal cost of area $F$. But importantly she weighs home and foreign variables differently. This difference explains a relevant mechanism of the model. For the sake of simplicity, consider the case in which $Y = Y_b$ (i.e. $\hat{\tau} = \tilde{\tau}_b$ and $\zeta_b = 0$). In contrast with what would happen in a first best world, under this parametric restriction, the target of common central bank of area $F$ reacts only to foreign markup shocks and is instead unaffected by home markup shocks. The reason is the following. By overlooking the external effects on the welfare of the households living in the other area, policymakers of the monetary union want to make them work more and pay more for domestic goods\textsuperscript{37}. Differently from small open economy policymakers, they realize that an improvement of their terms of trade can be generated by either decreasing domestic output or increasing foreign output. As a consequence, at least under the baseline calibration, they consider the foreign steady state output too low. Thus, they are not willing to stabilize foreign markup shocks, as would be Pareto efficient, because in this way the foreign per-period output can increase bringing about welfare benefits for domestic consumers.

The optimal monetary policy problem of the common central bank in area $H$ can be formulated as maximizing (73) with respect to $\tilde{y}_{H,t}^{b}, \tilde{y}_{F,t}^{b}, c_{H,t}^{b}, c_{F,t}^{b}$ and $\pi_{H,t}$ subject to the following sequence of sequence of constraints:

$$\tilde{y}_{H,t}^{b} = c_{H,t}^{b} + (1 - \delta_b)(c_{F,t}^{b} - c_{H,t}^{b})$$
$$\tilde{y}_{F,t}^{b} = c_{F,t}^{b} + (1 - \delta_b)(c_{H,t}^{b} - c_{F,t}^{b})$$
$$\pi_{H,t} = \lambda \left[ \tilde{\nu}_{H,t}^{b} + \sigma \left( c_{H,t}^{b} + (1 - \gamma_b)(c_{F,t}^{b} - c_{H,t}^{b}) \right) \right] + v_{H,t}^{b} + \beta E_t[\pi_{H,t+1}]$$
$$\pi_{F,t} = \lambda \left[ \tilde{\nu}_{F,t}^{b} + \sigma \left( c_{F,t}^{b} + (1 - \gamma_b)(c_{H,t}^{b} - c_{F,t}^{b}) \right) \right] + v_{F,t}^{b} + \beta E_t[\pi_{F,t+1}]$$

for all $t$ where $\tilde{\nu}_{t}^{b} = \lambda \left( \tilde{\nu}_{H,t}^{b} + \sigma \tilde{\nu}_{H,t}^{b} + \sigma (1 - \gamma_b)(c_{F,t}^{b} - c_{H,t}^{b}) - (1 + \varphi)\hat{\alpha}_{H,t} + \hat{\mu}_{H,t} \right)$, and $\tilde{\nu}_{F,t}^{b} = \lambda \left( \tilde{\nu}_{F,t}^{b} + \sigma \tilde{\nu}_{F,t}^{b} + \sigma (1 - \gamma_b)(c_{H,t}^{b} - c_{F,t}^{b}) - (1 + \varphi)\hat{\alpha}_{F,t} + \hat{\mu}_{F,t} \right)$. The solution to this problem allows to determine the average union home inflation and all the other average area variables, given a state contingent path of the average union foreign inflation. A symmetric problem can be stated for the foreign area. Notice that once the average union variables are determined, the region specific variables can be recovered directly from the equilibrium conditions namely (45), (46), (59) (60), (32) and (33). Moreover, under this formulation, the optimal monetary policy problem is independent of whether there is either monetary autonomy among countries or a monetary union in the other area.

\textsuperscript{37}...at least under the baseline calibration. In fact, in order to render the target independent of foreign markup shocks, the labour subsidy should be set equal to $\hat{\tau} = -\frac{(1 - \delta_b)(\sigma + \varphi)}{(1 - \delta_b)(\sigma + \varphi)}, a level such that foreign labour is over-subsidized.
5 Optimal monetary policies

The solution to the optimal policy problems of both the small open policymaker and the central bank of the monetary union enable us to simulate the impulse responses to a one percentage decrease in home and foreign markups under regimes A and B. These impulses responses are plotted in figures 1-3. The baseline calibration is listed in the appendix and is in line with the literature\textsuperscript{38}.

5.1 Dynamic Simulation

The impulse responses to a global negative markup shock can be interpreted as follows. As shown in figures 1 and 2, under optimal policies, given the decrease in the marginal costs, both home and foreign firms cut prices and expand output supply. Workers increase consumption and reduce leisure. Monetary policies have then to trade off between output and inflation stabilization. These patterns are common to both areas and regimes. However, under regime A consumption and output in area $H$ increase by less than in regime $B$, while deflation in area $H$ and output in area $F$ increase by more. These differences are explained \textit{exclusively} by the diverging conduct of policymakers under the two policy regimes.

Under regime $B$, when there are two currency unions, impulse responses are symmetric across areas. In this case, under markup shocks, both monetary authorities of areas $H$ and $F$ weight different incentives. The first incentive stems from the fact that under the baseline calibration domestic steady state output is too low from their perspective. This changes the inflation and output trade-off with respect to the case in which the steady state is at their desired level. Thus, in response to a domestic negative markup shock, monetary policymakers are willing to let domestic output and consumption increase and to focus more on the stabilization of domestic price dynamics. The second incentive arises from the fact that also foreign output and consumption are perceived as too low. As a consequence, the central banks of the monetary union favour the increase in foreign production due to a negative foreign markup shock. These two incentives, do not exactly balance. Indeed, under a global negative markup shock, the incentive to improve their terms of trade in order to allow foreign output to increase more than the domestic one prevails. Obviously, given the symmetry, in equilibrium none is able to influence the terms of trade.

Under regime $A$, the conduct of the monetary policymakers in area $H$ is dissimilar from that in regime $B$ in two respects. On the one hand, under the baseline calibration, the domestic steady state output is too high from the small open economy policymakers perspective. As a result, when there is monetary independence, they are more focused on output stabilization than the central bank of the monetary union and adopt a more countercyclical monetary policy. Indeed, in response to a negative markup shock, they seek to restrain output expansion\textsuperscript{39} and allow for a higher deflation by increasing on average the nominal interest rate by more than what the single central bank of the monetary union does in regime $B$. Notice that because of this incentive the independent


\textsuperscript{39}At the margin they find it optimal to reduce more consumption than leisure because of the steady state distortion.
monetary policymakers push the economy in the direction of an improvement of the terms of trade of their area. On the other hand, being their economies small, monetary authorities consider what happens in the world economy as exogenous. Thus, they do not realize (as the monetary authority of a currency area does) that if they increase the foreign per-period output relative to domestic per-period output, foreign markup shocks can be beneficial for domestic consumers.

Given the restrictive monetary policy in area $H$, the monetary authority of area $F$ restrains monetary policy as well, but not as much as the central banks of area $H$, allowing for a terms of trade worsening. By doing so, she wants to oppose the countercyclical policies of the countries of the other area, because she finds an expansion of foreign output beneficial. Nevertheless, she also wants to stabilize domestic price dynamics. Deflation response in area $F$ is similar across regimes, whereas output and consumption are influenced by the restrictive policy of the policymakers in area $H$.

There is a crucial question that is still left open. When are the consumers of area $H$ better off? Either in regime $A$ or in regime $B$? This question is addressed in the next section. However, for that analysis it is worth to notice the following. Given a negative markup shock, under regime $A$, the incentives of both the small open economies and the monetary union policymakers ensure an improvement of their terms of trade. As emphasized by figure 3, the impulse responses say that in equilibrium area $F$ experiences a terms of trade worsening. This result apparently goes against the insight that in regime $A$ the policymaker of the monetary union in area $F$ has more monopoly power in affecting the terms of trade across areas. Nevertheless, if we consider the terms of trade in terms of gaps from the targets, the picture changes. The terms of trade gap of the small open economy authorities is larger than that of monetary authority of the monetary union. Indeed, if target of the small open economy authority were implement, the consequent improvement of the terms of trade across areas would be much stronger than that implied by the target of the central bank of the monetary union in area $F$.

6 Welfare evaluation

The analysis of the previous section reveals that, in the presence of markup shocks, there are potential welfare benefits from the adoption of a common currency. Moreover, it makes clear which are the sources of these benefits: on the one hand the internalization of the spillover effects generated within area $H$; on the other hand the gains in monopoly power on the terms of trade across areas. The household welfare based criterion derived in (73) allows to quantify the welfare gains of being in a currency area as average per-period losses expressed as a fraction of the steady state consumption. The results are quite robust: under markup shocks, even for relatively low levels of the elasticity of substitution between home and foreign bundles, there are welfare benefits of forming in a monetary union. In the next sections we analyze how these benefits vary according to the key parameters of the model.
6.1 The intertemporal and the intratemporal elasticities of substitution

Both the intertemporal elasticity of substitution between home and foreign bundles, \( \eta \), and the relative risk aversion coefficient (the inverse of intertemporal elasticity of substitution of consumption), \( \sigma \), are crucial to determine the size of the welfare gains (or losses) of abandoning monetary autonomy\(^{40}\). Indeed they influence directly the effects that movements in the terms of trade produce on the demand and the supply of domestic goods. The higher are \( \eta \) and \( \sigma^{41} \), the larger is the switching effect from domestic towards foreign goods due to a terms of trade improvement. The higher is \( \sigma \), the larger is the decrease in firms real marginal cost due to an appreciation in the terms of trade. In synthesis, these parameters govern the real effects of the beggarthy-neighbour policies and therefore the benefits of policy coordination that arise from being in a monetary union. Figure 5 plots how welfare benefits increase in area \( H \) relatively to an increase of \( \eta \) and \( \sigma \). \( \eta \) varies from 1 to 3, while \( \sigma \) varies from 1 to 2.5. Within this range, these gains reach a maximum of 0.3 percentage of the steady state consumption. However, for low levels \( \eta \) the adoption of a common currency ensues welfare losses up to 0.1 percentage of steady state consumption.

6.2 The degree of home bias

We have already emphasized that the welfare benefits of a monetary union are due to two main channels: the internalization of all the external effects produced within the monetary union by the national authorities; the gains of monopoly power on the terms of trade across areas. A relevant question is which of these channels contributes more to explain the welfare benefits themselves. For this reason, we investigate to what extent these gains depend on the degree of home bias of area \( H, \alpha_h \).

Figure 5 plots the welfare gains of being in the regime \( B \) for the consumers of area \( H \) relatively to different degree of \( \eta \) (from 1 to 3) and \( \alpha_h \) (from 0.6 to 1) and shows the following result. For low degree of \( \eta \) the welfare gains\(^{42}\) are lower in a closed economy (i.e. \( \alpha_h = 1 \)), whereas for high degree of \( \eta \) the converse is true. This finding can be explained by recalling that \( \eta \) is at the same time the elasticity of substitution between the goods produced in a small economy and in the rest of world and between the bundles produced in the two areas. The lower is this elasticity, the higher is the degree of monopoly power that policymakers hold on the terms of trade across areas. The higher is this elasticity, the stronger are the spillover effects generated by beggarthy-neighbour policies\(^{43}\). Hence, when \( \eta \) is low, the main benefit of sharing a common

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\(^{40}\)This finding is actually consistent with the literature. See in particular Benigno and Benigno (2003), Benigno and Benigno (2006) and Pappa (2004).

\(^{41}\)The lower is the intertemporal elasticity of consumption, the higher is the incentive to smooth consumption across periods. Thus, when there is a terms of trade improvement, consumers are more inclined to keep the same level of overall consumption, buying more foreign goods or working more to substitute between the present and future consumption.

\(^{42}\)...which are actually losses.

\(^{43}\)In this case, the presence of area \( F \) can produce positive excentualities that outweigh the negative externality due to the monopoly power on the terms of trade across areas. Indeed, by taking into account the impact of her on the world economy, the the central banker of area \( F \) pushes the economies of area \( A \)
currency is due to the gain in monopoly power on the terms of trade. While when $\eta$ is high, the main benefit derives from avoiding the spillover effects generated within the area when there is monetary autonomy.

In order to better disentangle these two sources of welfare gains, it would be useful to allow for different elasticities of substitution between bundles produced in different regions and between bundles produced in different areas. In this way, in fact, it would be possible to understand how the welfare gains of forming a monetary union vary in response to a variation of a parameter, the elasticity of substitution between bundles produced in different areas, that affects exclusively the degree of monopoly power of the big economy on the terms of trade across areas.

6.3 The variance of the idiosyncratic shocks

For the purpose of this paper, it is important to check how welfare gains depend on the variance of the idiosyncratic of the shocks. Indeed, the costs of adopting a common currency are due to the loss of the an independent instrument of policy that can suit specific country economic conditions.

Figure 6 plots the welfare gains of the consumers in area $H$ relative to the elasticity of substitution between home and foreign bundles $\eta$ and to the ratio between the variance of the idiosyncratic innovations and that of average innovations $\zeta$. At a first sight, this figure is a bit puzzling. For high values of $\eta$, the welfare gains of being in a monetary union are increasing in $\zeta$, whereas for low values $\eta$ the converse is true. The key insight is the following: given the high values of $\eta$, national monetary policies generate such distortional effects that they outweigh the benefits of having a policy instrument that can stabilize country specific shocks. This intuition seems to be corroborated by the fact that, when $\eta$ is sufficiently low, the welfare gains decrease with the increase in $\zeta$.

7 Conclusion

This paper has shown that, in the presence of markup shocks, under plausible calibration there are welfare gains due to the adoption of a common currency. This finding is obtained in a New Keynesian open economy framework in which forming a monetary union entails a meaningful trade-off: on the one hand because of nominal rigidities, losing monetary independence implies the welfare costs of renouncing to a policy instrument that can stabilize country-specific shocks; on the other hand, delegating the monetary policy to the monetary union’s central bank generates welfare gains thank to the improvement upon the conduct of the national authorities. In a world constituted by two economic areas as the one laid out in our basic setup, two are the main sources of this improvement. The first is due to the internalization of the spillover effects produced by autonomous authorities within the monetary union. The second is ensued by the gain in monopoly power on the terms of trade across areas that balances that of the policymaker of the other big economy.

towards a more efficient performance. This is clearly understood once we recall that because of risk sharing consumption is highly correlated across areas.
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### Baseline Calibration

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Figure 1: Impulse responses to a negative aggregate markup shock.
Figure 2: Impulse responses to a negative aggregate markup shock.
Figure 3: Impulse responses to a negative aggregate markup shock; up on the right handside there is the plot of the terms of trade gaps under regime $A$ and on the left handside that of the terms of trade gap under regime $B$. 

Figure 4: Welfare gains for area H expressed as percentage of the steady state consumption.
Figure 5: Welfare gains for area H expressed as percentage of the steady state consumption.
Figure 6: Welfare gains for area H expressed as percentage of the steady state consumption.
A Retriving condition (26)
Given the definitions of $P_{H,t}^*$ and $P_{F,t}^*$ it is easy to show that:

$$
\mathcal{E}_{iH,t}P_{H,t}^* = [\alpha_b P_{H,t}^{1-\eta} + (1-\alpha_b) P_{F,t}^{1-\eta}]^{\frac{1}{1-\eta}} \quad \mathcal{E}_{iF,t}P_{F,t}^* = [\alpha_b P_{F,t}^{1-\eta} + (1-\alpha_b) P_{H,t}^{1-\eta}]^{\frac{1}{1-\eta}}
$$

By (75):

$$
\frac{\mathcal{E}_{iH,t}P_{H,t}^*}{P_{H,t}} = \left[ \alpha_b + (1-\alpha_b) \left( \frac{P_{F,t}}{P_{H,t}} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}} \quad \frac{\mathcal{E}_{iF,t}P_{F,t}^*}{P_{F,t}} = \left[ \alpha_b + (1-\alpha_b) \left( \frac{P_{H,t}}{P_{F,t}} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}}
$$

which jointly with (25) leads to:

$$
\left( \frac{C_{F,t}^*}{C_{H,t}} \right) = \left[ \frac{\alpha_b \left( \frac{P_{F,t}}{P_{H,t}} \right)^{1-\eta} + (1-\alpha_b)}{(1-\alpha_b) \left( \frac{P_{F,t}}{P_{H,t}} \right)^{1-\eta} + \alpha_b} \right]^{-\frac{1}{\sigma(1-\eta)}}
$$

Moreover thanks to (6):

$$
\frac{P_{i,t}}{P_{C_i,t}} = \left[ \frac{1}{\alpha_s} - \frac{\alpha_b - \alpha_s}{\alpha_s} \left( \frac{P_{H,t}}{P_{C_i,t}} \right)^{1-\eta} - (1-\alpha_b) \left( \frac{P_{F,t}}{P_{C_i,t}} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}} \quad i \in \left[ 0, \frac{1}{2} \right]
$$

which can be read as:

$$
\frac{P_{i,t}}{P_{C_i,t}} = \left[ \frac{1}{\alpha_s} - \frac{\alpha_b - \alpha_s}{\alpha_s} \left( \frac{P_{H,t}}{P_{C_i,t}} \right)^{1-\eta} \left( \frac{C_{H,t}^*}{C_i} \right)^{-\sigma(1-\eta)} - (1-\alpha_b) \left( \frac{P_{F,t}}{P_{C_i,t}} \right)^{1-\eta} \left( \frac{C_{F,t}^*}{C_i} \right)^{-\sigma(1-\eta)} \right]^{\frac{1}{1-\eta}}
$$

Finally by using (76) and (77) we can rewrite (78) as (26)

B Zero Inflation Deterministic Steady State

In this section we show that, given appropriate initial conditions, under both regimes, $A$ and $B$, at the deterministic steady state, zero inflation is a Nash equilibrium policy.

In the regime $A$ the timelessly optimal policy problem of a monetary authority of country $i$ in the area $H$ can be formulated as the maximization of the following Lagragian:

32
\[
L^i = \sum_{t=0}^{\infty} \beta^t E_0 \left\{ \frac{C_t^{i-\sigma}}{1-\sigma} - \frac{1}{\varphi + 1} \left( \frac{Y_t^i Z_t^i}{A_t^i} \right)^{\varphi+1} \right\} \\
+ \zeta_{s,i}^2 \left[ Y_t^i \left( \frac{P_{t,i}}{P_{C^i,t}} \right)^{\eta} \left( \alpha_s C_t^i + (\alpha_b - \alpha_s) C_{F,t}^{i,\sigma} \right) + \frac{1}{\varphi + 1} \right] - \zeta_{s,i}^2 \theta \Pi_{s,t}^i K_t^i \\
+ \zeta_{s,i}^4 \left[ F_t^i - K_t^i \left( \frac{1 - \theta \Pi_{s,t}^{i-1}}{1 - \theta} \right)^{\tilde{\tau}} \right] - \zeta_{s,i}^4 \theta \Pi_{s,t}^{i-1} F_t^i \\
+ \zeta_{s,i}^5 \left[ Z_t^i - \theta Z_t^i \Pi_{s,t}^i - (1 - \theta) \left( \frac{1 - \theta \Pi_{s,t}^{i-1}}{1 - \theta} \right)^{\tilde{\tau}} \right] \right\}
\]

with respect to \( C_t^i, Y_t^i, Z_t^i, K_t^i, F_t^i \) and \( \Pi_{s,t}^i \) and where \( P_{t,i}/P_{C^i,t} \) are determined consistently with (26), while \( C_{H,t}^i, C_{F,t}^i, C_{H,t}^s \) and \( C_{F,t}^s \) are taken as given. Assume that \( \mu_t^i = \mu, A_t^i = A, \tau^j = \tau \) and \( Z_t^j = \Pi_{j,t}^i = 1 \) for all \( j \in [0,1] \) and \( t \). Assume in addition that \( Z_{t-1}^i = 1 \). Recalling that \( \tilde{\tau} = 1 - \frac{(1-\tau)(\epsilon-1)}{1+\mu} \), it can be shown that according to the first order conditions at the symmetric deterministic steady state:

\[
\begin{align*}
C^{-\sigma} &= \zeta_1^s \delta_s - \zeta_3^s \eta Y C^{-\sigma-1} \\
Y^{\varphi} &= \zeta_1^s + \zeta_2^s Y^{\varphi+1} + \zeta_3^s (1 - \theta) \\
Y^{\varphi+1} &= \zeta_2^s Y^{\varphi} + \zeta_3^s (1 - \theta) \\
\zeta_2^s (1 - 1) &= \zeta_3^s \\
\zeta_3^s (1 - 1) &= -\zeta_4^s \\
\zeta_2^s \theta \varepsilon K &= -\zeta_3^s \theta (\varepsilon - 1) F + \zeta_4^s \frac{\theta}{1 - \theta} K
\end{align*}
\]  

with \( \delta_s = \alpha_s (1 - \sigma \eta) + \gamma_s \eta \sigma \). Then

\[
Y = (1 - \tilde{\tau})^{-\frac{1}{1+\tilde{\tau}}} \\
C = Y \\
F = K = \frac{Y C^{-\sigma}}{1 - \theta} = \frac{Y^{\varphi+1} (1 - \tilde{\tau})}{1 - \theta} \\
\Pi_H = \Pi_F = 1 \quad Z = 1 \\
\zeta_1^s = \frac{\gamma_s \sigma + (1 - \tilde{\tau} \varphi)}{\delta_s \varphi + \gamma_s \sigma} \\
\zeta_2^s \frac{1}{1 - \theta} = -\zeta_3^s = -\frac{\delta_s - (1 - \tilde{\tau}) \varphi}{(\delta_s \varphi + \gamma_s \sigma)(1 - \tilde{\tau})} \quad \zeta_3^s = \frac{Y^{\varphi+1} (1 - \varphi \zeta_3^s)}{1 - \theta}
\]
is a steady state symmetric solution of the optimal policy problem just stated\(^{44}\).

Consider now the monetary union in the area \(F\). Suppose that for all \(i \in [0, \frac{1}{2})\) \(\Pi^i_t = 1\) at all times\(^{45}\). Then we want to show that given other policymakers strategy, \(\Pi^i_t = 1\) for all \(i \in [\frac{1}{2}, 1]\) and \(t\) is optimal.

If for all \(i \in [0, \frac{1}{2})\) \(\Pi^i_t = 1\) at all times, the optimal policy problem of the monetary authority in the area \(F\) can be written as maximizing:

\[
L = \sum_{t=0}^{\infty} \beta^t E_0 \left\{ \int_0^1 C^i_{t}^{1-\sigma} \left[ Y^i_t Z^i_t \right] \right\}
\]

\[
+ \zeta^i_{1,t} \left[ Y^i_t - \left( \frac{P_{i,t}}{P_{C,i,t}} \right)^{-\eta} \left( \alpha_s C^i_t + 2(\alpha_b - \alpha_s) C^i_t \int_0^1 C^j_t^{1-\sigma} dj + 2(1 - \alpha_b) C^i_t^{\eta} \int_0^1 C^j_t^{1-\eta} dj \right) \right] - \zeta^i_{2,t} \theta \Pi^i_{t-1} F^i_t
\]

\[
+ \zeta^i_{3,t} \left[ F^i_t - K^i_t \left( \frac{1 - \theta \Pi^i_{t-1}}{1 - \theta} \right)^{\frac{1}{\tau}} \right] - \zeta^i_{4,t} \left[ Z^i_t - \theta Z^i_{t-1} \Pi^i_{t-1} - \left( 1 - \theta \right) \left( \frac{1 - \theta \Pi^i_{t-1}}{1 - \theta} \right)^{\frac{1}{\tau}} \right]
\]

\[
+ \zeta^i_{5,t} \left[ \left( \frac{C_{F,i,t}}{C_{F,i,t-1}} \right)^{-\sigma} \frac{P_{F,i,t}}{P_{F,i,t-1}} \Pi^i_{t-1} F^i_t - \left( \frac{C^i_t}{C^i_{t-1}} \right)^{-\sigma} \frac{P_{i,t}}{P_{C,i,t}} \Pi^i_{t-1} F^i_t \right] \right\} \right\}
\]

\[
+ \zeta^i_{6,t} \left[ \left( \frac{C_{F,i,t}}{C_{F,i,t-1}} \right)^{-\sigma} \frac{P_{F,i,t}}{P_{F,i,t-1}} \Pi^i_{t-1} F^i_t - \left( \frac{C^i_t}{C^i_{t-1}} \right)^{-\sigma} \frac{P_{i,t}}{P_{C,i,t}} \Pi^i_{t-1} F^i_t \right] \right\}
\]

\[
+ \int_0^1 \left\{ \zeta^i_{7,t} \left[ Y^i_t - \left( \frac{P_{i,t}}{P_{C,i,t}} \right)^{-\eta} \left( \alpha_s C^i_t + 2(\alpha_b - \alpha_s) C^i_t \int_0^1 C^j_t^{1-\sigma} dj + 2(1 - \alpha_b) C^i_t^{\eta} \int_0^1 C^j_t^{1-\eta} dj \right) \right] \right\}
\]

\[
+ \zeta^i_{8,t} \left[ \left( 1 + \mu^i_t \right) (1 - \tau) \left( Y^i_t \right)^{\varphi+1} - \frac{P_{i,t}}{P_{C,i,t}} Y^i_t C^i_t^{-\sigma} \right] \right\} \right\}
\]

with respect to \(C^i_t, Y^i_t\) for all \(i\) and \(Z^i_t, K^i_t, F^i_t\) and \(\Pi^i_t\) all \(i \in [\frac{1}{2}, 1]\) and where \(P_{i,t}/P_{C,i,t}\) and \(P_{F,i,t}/P_{F,i,t}\) are determined consistently with \((26), (27), (76)\) and \((77)\).

Assume that \(\mu^i_t = \mu, A^i_t = A, \tau^i = \tau\) and \(Z^i_t = \Pi^i_{t-1} = 1\) for all \(j \in [0, 1]\) and \(t\).

Moreover assume that \(Z^i_{t-1} = 1\) for all \(i \in [\frac{1}{2}, 1]\) Given that \(\hat{\tau} = 1 - \frac{(1-\tau)}{(1+\mu) \sqrt{\tau}}\). Then

\[^{44}\text{In other words, given a zero inflation policy of the other central banks, zero inflation is a best response of the central bank of the country } i.\]

\[^{45}\text{We follow closely Benigno and Benigno (2006).}\]

\[^{46}\text{which implies that } F^i_t = F, K^i_t = K \text{ and } F^i_t, K^i_t = 1 \text{ for all } i \text{ and } t.\]
according to the first order conditions at the symmetric deterministic steady state:

\[ C^{-\sigma} = \zeta_1^b \delta_b + \zeta_2^b (1 - \delta_b) - \zeta_3^b \sigma \gamma_b Y C^{-\sigma - 1} - \zeta_8^b \sigma (1 - \gamma_b) Y C^{-\sigma - 1} \quad (86) \]

\[ Y^\varphi = \zeta_1^b - \zeta_2^b (\varphi + 1) Y^\varphi (1 - \bar{\tau}) - \zeta_3^b C^{-\sigma} \quad (87) \]

\[ Y^{\varphi + 1} = -\zeta_2^b \varphi Y^{\varphi + 1} + \zeta_3^b (1 - \theta) \quad (88) \]

\[ \zeta_2^b (1 - \theta) = \zeta_4^b \quad (89) \]

\[ \zeta_3^b (1 - \theta) = -\zeta_4^b \quad (90) \]

\[ \zeta_3^b \theta \varepsilon K = -\zeta_3^b \theta (\varepsilon - 1) F + \zeta_4^b \frac{\theta}{1 - \theta} K \quad (91) \]

\[ 0 = \zeta_1^b (1 - \delta_b) + \zeta_2^b \delta_b - \zeta_3^b \sigma (1 - \gamma_b) Y C^{-\sigma - 1} - \zeta_8^b \sigma \gamma_b Y C^{-\sigma - 1} \quad (92) \]

\[ 0 = \zeta_7^b + \zeta_8^b \left[ (\varphi + 1) Y^\varphi (1 - \bar{\tau}) - C^{-\sigma} \right] \quad (93) \]

where \( \delta_b \equiv (1 - \sigma \eta) \alpha_b + \eta \sigma \gamma_b \).

Then it is easy to show:

\[ Y = (1 - \bar{\tau})^{-\frac{1}{\sigma + \varphi}} \quad (94) \]

\[ C = Y \quad (95) \]

\[ F = K = \frac{Y C^{-\sigma}}{1 - \theta} = \frac{Y^{\varphi + 1} (1 - \bar{\tau})}{1 - \theta} \quad (96) \]

\[ \Pi_H = \Pi_F = 1 \quad Z = 1 \quad (97) \]

Hence being the best response of both monetary union and the small open economy policymakers, zero inflation is a Nash equilibrium solution in regime \( A \).
Consider now the case of regime $B$ and suppose that the central bank of area $H$ set $\Pi_{H,t}^{-1} = 1$ for all $t$. The central bank of the monetary union in area $F$ maximizes:

$$L = \sum_{t=0}^{\infty} \beta^t E_0 \left\{ \int_0^1 \left[ C_i^{1-\sigma} - \frac{1}{\varphi + 1} \left( Y_i^i Z_i^i \right)^{\varphi+1} \right] + \sum_{i,t}^b \left[ Y_i^i - \left( \frac{P_{F,t}}{P_{C,t}} \right)^{-\eta} \left( \alpha_i C_t^{\sigma} + 2(\alpha_b - \alpha_i) C_t^{\sigma \eta} \right) \int_0^1 C_t^{1-\sigma \eta} dj + 2(1 - \alpha_b) C_t^{\sigma \eta} \int_0^1 C_t^{1-\sigma \eta} dj \right] + \sum_{i,t}^H \left[ K_i^i - \left( \frac{C_{H,t}}{C_{F,t}} \right)^{-\sigma} P_{H,t}^{*} \frac{P_{F,t}^{*}}{P_{C,t}^{*}} \Pi_{H,t}^{*} - \left( \frac{C_{F,t}}{C_{t}} \right)^{-\sigma} P_{t}^{*} \frac{P_{C,t}^{*}}{P_{t}^{*}} \Pi_{t}^{*} \right] \right\} di$$

with respect to $C_i^i$, $Y_i^i$, $Z_i^i$, $K_i^i$, $F_i^i$ and $\Pi_{i,t}$ all $i$ and where $P_{t}^{*}/P_{C,t}^{*}$, $P_{F,t}^{*}/P_{C,t}^{*}$ and $P_{H,t}^{*}/P_{H,t}^{*}$ are determined consistently with (26), (27), (76) and (77). Assume that

$$\mu_i = \mu, A_i = A, \tau_i = \tau \text{ and } Z_i = \Pi_{j,t} = 1 \text{ for all } j \in [0,1] \text{ and } t. \text{ Moreover assume that } Z_i^{-1} = 1 \text{ for all } i. \text{ Given that } \bar{\tau} = 1 - \frac{(1-\mu_i)}{(1-\tau_i)}, \text{ then according to the first order conditions at the symmetric deterministic steady state it can be shown:}$$
\[ Y = (1 - \hat{\tau})^{-\frac{1}{\sigma}} \]  
\[ C = Y \]  
\[ F = K = \frac{Y C^{-\sigma}}{1 - \theta} = \frac{Y^{\varphi+1}(1 - \hat{\tau})}{1 - \theta} \]  
\[ \Pi_H = \Pi_F = 1 \quad Z = 1 \]

Therefore for the policymaker of the area \( F \) zero inflation is a best response to a zero inflation policy of the policymaker in the area \( H \). A symmetric problem can be stated for the policymaker of the monetary union of the area \( H \) Thus zero inflation is a Nash equilibrium policy.

\section*{C The purely quadratic approximation of the welfare}

In order to recover the optimal policies we need to approximate up to the second order single country representative agent utility given by (1) in the following way.

First we can approximate the utility derived from private consumption for generic region \( i \) as:

\[ \frac{C_i^{1-\sigma}}{1-\sigma} \simeq \frac{C_i^{1-\sigma}}{1-\sigma} + C_i^{1-\sigma}(c_i^{\hat{c}_i} + \frac{1}{2}(c_i^{\hat{c}_i})^2) - \frac{\sigma}{2} C_i^{1-\sigma}(c_i^{\hat{c}_i})^2 + t.i.p. \quad (102) \]

where \( c_i^{\hat{c}_i} \) stands for the log-deviations of private consumption from the non-stochastic symmetric steady state\(^{47}\).

Similarly the labor disutility can be approximated by taking into account that \( N_i = \frac{Y_i Z_i T}{A_i} \) and, as showed by Gali and Monacelli (2005), being \( Z_i = \int_0^1 \left( \frac{m(t)}{Y_i t} \right)^{-\varepsilon} dh^i \):

\[ \hat{z}_i^t \simeq \frac{\varepsilon}{2} Var_h(p_i(h^i)) \quad (103) \]

In words the approximation of \( Z_i^t \) around the symmetric steady state is purely quadratic. Moreover following Woodford (2001, NBER WP8071) it is possible to show that

\[ \sum_{t=0}^{\infty} \beta^t Var_{h^i}(p_i(h^i)) = \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \pi_{t,\theta}^2 \] with \( \lambda = \frac{(1-\theta)(1-\beta)}{\theta} \). Thus:

\[ \frac{1}{\varphi + 1} \left( \frac{Y_i Z_i T}{A_i} \right)^{\varphi+1} \simeq \frac{1}{\varphi + 1} Y^{\varphi+1} + Y^{\varphi+1}(\hat{y}_i^t + \frac{1}{2}(\hat{y}_i^t)^2) + Y^{\varphi+1} \varepsilon (\pi_{t,\theta})^2 + \frac{\varphi}{2} Y^{\varphi+1}(\hat{y}_i^t)^2 \]

\[ - (\varphi + 1) Y^{\varphi+1} \hat{y}_i^t a_i^t + t.i.p. \quad (104) \]

\(^{47}\)From now this convention will be used: \( \hat{x}_t \) represents the log-deviation of \( X_t \) from the steady state.
C.1 The case of the small open economy

By combining \( (102) \) and \( (104) \) and taking into account that at the steady state \( C^{-\sigma} = (1 - \tilde{\tau}) Y^f \), the second order approximation of welfare of the region \( i \) households can be written as:

\[
\sum_{t=0}^{\infty} \beta^t Y^f e^{t+1} E_0 \left[ s_i^t w_s - \frac{1}{2} \hat{s}_i^t W_{s,s} \hat{s}_i^t + \hat{s}_i^t W_{s,e} \hat{\delta}_i^t \right] + t.i.p. \tag{105}
\]

where

\[
\hat{s}_i^t \equiv [\hat{g}_i^t, \hat{\xi}_i^t, \pi_{i,t}] \quad w_s' \equiv [-1, (1 - \tilde{\tau}), 0] \quad \hat{\delta}_i^t \equiv [\hat{c}_{H,t}, \hat{c}_{F,t}, \eta_i, \mu_i]
\]

\[
W_{s,s} \equiv \begin{bmatrix}
(\varphi + 1) & 0 & 0 \\
0 & (1 - \tilde{\tau})(\sigma - 1) & 0 \\
0 & 0 & \hat{\xi}
\end{bmatrix} \quad W_{s,e} \equiv \begin{bmatrix}
0 & 0 & (\varphi + 1) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and with \( i \in [0, \frac{1}{2}] \) \( \hat{c}_{H,t} \equiv 2 \int_0^{\frac{1}{2}} \hat{c}_t^j dj \) and \( \hat{c}_{F,t} \equiv 2 \int_{\frac{1}{2}}^1 \hat{c}_t^j dj \). In order to recover a purely quadratic approximation to the welfare for the central bank of the small open economy, we have to use both the second order approximation to the demand and to the Phillips curves.

The second order approximation to the demand curve can be written as:

\[
0 \simeq \left[ \hat{s}_i^t g_s - \hat{\delta}_i^t g_e + \frac{1}{2} \hat{s}_i^t G_{s,s} \hat{s}_i^t - \hat{s}_i^t G_{s,e} \hat{\delta}_i^t \right] + s.o.t.i.p. \tag{106}
\]

where

\[
g_s' \equiv [-1, \delta_s, 0] \quad g_e' \equiv [-(\delta_b - \delta_s), -(1 - \delta_b), 0, 0]
\]

\[
G_{s,s} \equiv \begin{bmatrix}
0 & 0 & 0 \\
0 & \delta_s + \omega_1 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad G_{s,e} \equiv \begin{bmatrix}
0 & 0 & 0 & 0 \\
\omega_1 + \omega_2 & -\omega_2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

where \( \delta_s \equiv \alpha_s (1 - \eta \sigma) + \eta \sigma / \alpha_s, \delta_b \equiv \alpha_b (1 - \eta \sigma) - \alpha_b \eta \sigma / (1 - 2\alpha_b) \) and:

\[
\omega_1 = \frac{1 - \alpha_s (1 - \eta \sigma)}{\alpha_s^2} (\sigma - (1 - \alpha_s) \alpha_s (1 - \eta \sigma)) \quad \omega_2 = \frac{1 - \alpha_b (1 - \eta \sigma)}{\alpha_s (1 - 2\alpha_b)} (1 - \eta \sigma)
\]

As in Benigno and Woodford (2005) the second order approximation to the (54) and be combined with (52) and (53) to obtain:

\[
V_0 = \frac{1 - \theta}{\theta} (1 - \beta \theta) \sum_{t=0}^{\infty} \beta^t E_0 \left[ s_i^t v_s - \hat{\delta}_i^t v_e + \frac{1}{2} \hat{s}_i^t V_{s,s} \hat{s}_i^t - \hat{s}_i^t V_{s,e} \hat{\delta}_i^t \right] + s.o.t.i.p. \tag{107}
\]
where
\[ v'_s \equiv [\varphi, \sigma \gamma_s, 0] \quad v'_e \equiv [\sigma(\gamma_s - \gamma_b), -\sigma(1 - \gamma_b), -(\varphi + 1), 1] \]

\[
V_{s,s} \equiv \begin{bmatrix}
\varphi(\varphi + 2) & \sigma \gamma_s & 0 \\
\sigma \gamma_s & -\sigma^2 \gamma_s^2 & 0 \\
0 & 0 & \varphi(\varphi + 1) / \lambda
\end{bmatrix}
\]

\[
V_{s,e} \equiv \begin{bmatrix}
\sigma(\gamma_s - \gamma_b) & -\sigma(1 - \gamma_b) & (\varphi + 1) & -(\varphi + 1) \\
\sigma^2 \gamma_s(\gamma_b - \gamma_s) & \sigma^2 \gamma_s(1 - \gamma_b) & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Given (106) and (107), it is possible to rewrite (105) in a purely quadratic way. Indeed thanks to these conditions:

\[
0 \simeq Y^{\varphi + 1} \sum_{t=0}^{\infty} \beta^t E_0 \left[ \ddot{\hat{v}}' t (1 - \varphi \zeta_s) f_s - \zeta_s v_s + \frac{1}{2} \hat{s}_t^s ((1 - \varphi \zeta_s) F_{s,s} - \zeta_s V_{s,s}) \hat{s}_t^s - \hat{s}_t^e ((1 - \varphi \zeta_s) F_{s,e} - \zeta_s V_{s,e}) e_t^e \right] + t.i.p. \tag{108}
\]

where \( \zeta_s = (\delta_s - (1 - \bar{\tau})) / (\delta_s \varphi + \gamma_s \sigma) \). Notice that \( \zeta_s^3 = \zeta_s(1 - \bar{\tau}) \) and \( \zeta_s^4 = (1 - \varphi \zeta_s) \) with \( \zeta_s^3 \) and \( \zeta_s^4 \) being the lagrange multipliers previously recovered for the optimal policy problem of the small economy policymaker\(^{48}\). It is easy to show that:

\[
w_s = (1 - \varphi \zeta_s) f_s - \zeta_s v_s \tag{109}
\]

Hence we can write the second order approximation of union welfare as:

\[
Y^s \sum_{t=0}^{\infty} \beta^t E_0 \left[ -\frac{1}{2} \hat{s}_t^s \Omega_{s,s} \hat{s}_t^s + \hat{s}_t^e \Omega_{s,e} e_t^e \right] + t.i.p. \tag{110}
\]

where

\[
\Omega_{s,s} \equiv W_{s,s} + (1 - \varphi \zeta_s) G_{s,s} - \zeta_s V_{s,s} \quad \Omega_{s,e} \equiv W_{s,e} + (1 - \varphi \zeta_s) G_{s,e} - \zeta_s V_{s,e}
\]

and \( \Omega_{s,s} \) and \( \Omega'_{s,e} \) are respectively equal to:

\[
\begin{bmatrix}
(1 - \zeta_s(\varphi + 1)){\varphi} & -\zeta_s \gamma_s \sigma \\
-\zeta_s \gamma_s \sigma & (1 - \bar{\tau})(\sigma - 1) + \zeta_s \gamma_s^2 \sigma^2 + (1 - \zeta_s \varphi)(\delta_s + \omega_1) \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\zeta_s \sigma(\gamma_s - \gamma_b) & \zeta_s \sigma^2 \gamma_s(\gamma_s - \gamma_b) + (1 - \zeta_s \varphi)(\omega_1 + \omega_2) \\
\zeta_s \sigma(1 - \gamma_b) & -\zeta_s \sigma^2 \gamma_s(1 - \gamma_b) - (1 - \zeta_s \varphi) \omega_2 \\
(1 - \zeta_s(\varphi + 1))(\varphi + 1) & 0 \\
\zeta_s(\varphi + 1) & 0
\end{bmatrix}
\]

\(^{48}\)See Benigno and Woodford (2005)
Now we would like to rewrite this approximation in terms of deviations from the target of the small open economy policymaker. It can be shown this target can be determined by maximizing (108) with respect to $\tilde{y}_t^i, \tilde{c}_t^i$ and $\pi_{t,t}$ taking as given the aggregate variables in the area $H$ and $F$ and subject to (45). According to the first order conditions of this problem:

\[
(1 - \zeta_s(\varphi + 1))\varphi \tilde{y}_t^{i,s} - \zeta_s \sigma \gamma_s - \zeta_s \sigma (1 - \gamma_b) \tilde{c}_{H,t} - \zeta_s \sigma (1 - \gamma_b) \tilde{c}_{F,t} - (1 - \zeta_s(\varphi + 1))(\varphi + 1)\tilde{d}_t^i - \zeta_s(\varphi + 1)\tilde{\mu}_t^i = \phi_1^i, 
\]

\[
(1 - \bar{\tau})(\sigma - 1) + \zeta_s \gamma_s^2 \sigma^2 + (1 - \zeta_s \varphi)(\delta_s + \omega_1)(\tilde{c}^{i,s}_t - \zeta_s \gamma_s \tilde{y}_t^{i,s} - \zeta_s \gamma_s (1 - \gamma_b) - (1 - \zeta_s \varphi)\omega_2)\tilde{c}_{H,t} + (\zeta_s \gamma_s (1 - \gamma_b) + (1 - \zeta_s \varphi)\omega_2)\tilde{c}_{H,t} = -\delta_s \phi_1^i, 
\]

\[
(1 - \zeta_s(\varphi + 1))\tilde{\varepsilon}_t^i \pi_{t,t} = 0 
\]

\[
\tilde{y}_t^{i,s} = \delta_s \tilde{c}^{i,s}_t + (\delta_b - \delta_s)\tilde{c}_{H,t} + (1 - \delta_b)\tilde{c}_{F,t} 
\]

for all $i \in \{0, \frac{1}{2}\}$ and where $\phi_1^i$ is the lagrange multiplier of (45). Notice that in the perspective of the small open monetary authority $\tilde{c}_{H,t}$ and $\tilde{c}_{F,t}$ are taken as exogenous.

Then it is easy to show that (110) can be rewritten as (69). Indeed it is sufficient to add and subtract the corresponding target in each terms of (110) and then use the fist order conditions just listed.

### C.2 The case of the Monetary Union

If in the area $H$ there is a Monetary Union, then the second order approximation of average welfare of the union household can be read as:

\[
\sum_{t=0}^{\infty} \tilde{g}^t \gamma^{\varphi + 1} \int_0^2 E_0 \left[ \tilde{s}_t^{w_s} - \frac{1}{2} \tilde{p}_t^{s_s} W_{s,s} \tilde{s}_t^i + \tilde{u}_t^{s_s} W_{s,s} \tilde{u}_t^i \right] dt + t.i.p. 
\]

\[
\tilde{s}_t^{w_s} \equiv [\tilde{y}_t, \tilde{c}_t, \pi_{t,t}] \quad w_s^{i} \equiv [-1, (1 - \bar{\tau}), 0] \quad \tilde{u}_t^{i} \equiv [a_t^i, \mu_t^i] 
\]

\[
W_{s,s} \equiv \begin{bmatrix} (\varphi + 1) & 0 & 0 \\ 0 & (1 - \bar{\tau})(\sigma - 1) & 0 \\ 0 & 0 & \frac{1}{\bar{\tau}} \end{bmatrix} \quad W_{s,s} \equiv \begin{bmatrix} (\varphi + 1) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} 
\]

A purely quadratic approximation to the welfare of the union households can be retrieved thanks to the second order approximations of the demand and of the supply curves.
The second order approximation to the demand curve of a generic region \( i \) in the area \( H \) can be read as:

\[
0 \simeq \hat{s}\int_{0}^{1} \hat{s}_i' dt' g_{S H} + \int_{0}^{1} \hat{s}_i' dt' g_{S F} + \frac{1}{2} \int_{0}^{1} \hat{s}_i' G_{S S} \hat{s}_i' dt + \frac{1}{2} \int_{0}^{1} \hat{s}_i' G_{S H, S H} \hat{s}_i' dt + \frac{1}{2} \int_{0}^{1} \hat{s}_i' G_{S F, S F} \hat{s}_i' dt \]

\[
+ \frac{1}{2} \int_{0}^{1} \hat{s}_i' G_{S H, S H} \int_{0}^{1} \hat{s}_i' dt + \frac{1}{2} \int_{0}^{1} \hat{s}_i' G_{S F, S F} \int_{0}^{1} \hat{s}_i' dt + \hat{s}_i' G_{S S} \int_{0}^{1} \hat{s}_i' dt + \hat{s}_i' G_{S H, S H} \int_{0}^{1} \hat{s}_i' dt + \hat{s}_i' G_{S F, S F} \int_{0}^{1} \hat{s}_i' dt \]

\[
+ \frac{1}{2} \int_{0}^{1} \hat{s}_i' G_{S H, S F} \int_{0}^{1} \hat{s}_i' dt + s.o.t.i.p. \quad (117)
\]

where

\[
g' \equiv [-1, \delta_s, 0] \quad g'_{S H} \equiv [0, 2(\delta_b - \delta_s), 0] \quad g'_{S F} \equiv [0, 2(1 - \delta_b), 0, 0] \]

\[
G_{S S} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta_s + \omega_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad G_{S F, S F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1 - \delta_b) + \omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
G_{S H, S H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\eta \sigma^2 (1 - \gamma_s^2) + (\delta_b - \delta_s) - (\omega_1 + \omega_3) & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
G_{S H, S F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & (\eta \sigma^2 (1 - \gamma_s^2) - \eta \sigma^2 \gamma_b (1 - \gamma_b) + 2 \omega_1 + 2 \omega_2 + \omega_3) & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
G_{S F, S F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & \omega_1 & 0 \end{bmatrix} \quad G_{S S} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\omega_1 + \omega_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

and where

\[
\omega_3 \equiv \frac{(1 - \alpha_b) \eta \sigma (\alpha + 2(1 - \alpha_b)(1 - \eta \sigma))}{1 - 2 \alpha_b}
\]

By integrating (117):

\[
0 \simeq \hat{h}' \int_{0}^{1} \hat{s}_i' dt' h_{S H} + \int_{0}^{1} \hat{s}_i' dt' h_{S F} + \frac{1}{2} \int_{0}^{1} \hat{s}_i' H_{S H, S H} \hat{s}_i' dt + \frac{1}{2} \int_{0}^{1} \hat{s}_i' H_{S F, S F} \hat{s}_i' dt \]

\[
+ \frac{1}{2} \int_{0}^{1} \hat{s}_i' H_{S H, S H} \int_{0}^{1} \hat{s}_i' dt + \frac{1}{2} \int_{0}^{1} \hat{s}_i' H_{S F, S F} \int_{0}^{1} \hat{s}_i' dt + \hat{s}_i' H_{S S} \int_{0}^{1} \hat{s}_i' dt + \hat{s}_i' H_{S H, S H} \int_{0}^{1} \hat{s}_i' dt + \hat{s}_i' H_{S F, S F} \int_{0}^{1} \hat{s}_i' dt + s.o.t.i.p.
\]

with

\[
h'_{S H} \equiv [-1, \delta_b, 0] \quad h'_{S F} \equiv [0, (1 - \delta_b), 0, 0] \]

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\[ H_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\eta\sigma^2(1 - \gamma_2^2) + \delta_b - \omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad H_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1 - \delta_b) + \omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ H_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & \eta\sigma^2(1 - \gamma_2^2) - \eta\sigma^2\gamma_b(1 - \gamma_b) + \omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad H_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & \eta\sigma^2\gamma_b(1 - \gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

A symmetric approximation can be stated for the resource constraints of the regions in the area \( F \) namely:

\[
0 \simeq + \int_0^1 \hat{s}_t^i di' f_{s_F} + \int_0^1 \frac{1}{2} \hat{s}_t^i di' F_{s_F,s_F} \hat{s}_t^i di + \frac{1}{2} \int_0^1 \frac{1}{2} \hat{s}_t^i F_{s_F,s_F} \hat{s}_t^i di + \frac{1}{2} \int_0^1 \frac{1}{2} \hat{s}_t^i F_{s_H,s_H} \hat{s}_t^i di + \frac{1}{2} \int_0^1 \frac{1}{2} \hat{s}_t^i F_{s_H,s_H} \hat{s}_t^i di + s.o.t.i.p.
\]

with \( f_{s_F} \equiv h_{s_H}, f_{s_H} \equiv h_{s_F}, F_{s_F,s_F} \equiv H_{s_H,s_H}, F_{s_H,s_H} \equiv H_{s_F,s_F}, F_{s_F,s_F} \equiv H_{s_H,s_H}, F_{s_H,s_H} \equiv H_{s_F,s_F} \) and \( F_{s_H,s_H} \equiv H_{s_F,s_F} \).

Conversely the second order approximation of the (54) for the area \( F \) can be obtained by combining (52) and (53):

\[
V_0 = \frac{1 - \theta}{\theta} (1 - \beta \theta) \sum_{t=0}^\infty \beta^t E_0 \left[ \hat{s}_t^i v_s + \int_0^1 \frac{1}{2} \hat{s}_t^i v_{s_F} + \int_0^1 \frac{1}{2} \hat{s}_t^i v_{s_H} - \hat{u}_t^i \right] + \frac{1}{2} \int_0^1 \frac{1}{2} \hat{s}_t^i V_{s,s} \hat{s}_t^i di
\]

\[
+ \frac{1}{2} \int_0^1 \frac{1}{2} \hat{s}_t^i V_{s,F} \hat{s}_t^i di + \frac{1}{2} \int_0^1 \frac{1}{2} \hat{s}_t^i V_{s,H} \hat{s}_t^i di + \frac{1}{2} \int_0^1 \frac{1}{2} \hat{s}_t^i V_{s,F} \hat{s}_t^i di + \frac{1}{2} \int_0^1 \frac{1}{2} \hat{s}_t^i V_{s,H} \hat{s}_t^i di + s.o.t.i.p.
\]

where

\[
v'_s \equiv [\varphi, \sigma \gamma_s, 0] \quad v'_F \equiv [0, 2\sigma (\gamma_b - \gamma_s), 0] \quad v'_H \equiv [0, 2\sigma (1 - \gamma_b), 0] \quad v'_u \equiv [ (\varphi + 1), -1] \]

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\[ V_{s,s} = \begin{bmatrix} \varphi(\varphi + 2) & \sigma \gamma_s & 0 \\ \sigma \gamma_s & -\sigma^2 \gamma_s^2 & 0 \\ 0 & 0 & \varepsilon(\varphi + 1) \end{bmatrix} \quad V_{S_F,S_F} = \begin{bmatrix} 0 & 0 & \sigma^2(\gamma_b - \gamma_s)^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ V_{S_H,S_H} = \begin{bmatrix} 0 & 0 & \sigma(\gamma_b - \gamma_s) & 0 \\ 0 & 0 & -\sigma^2(\gamma_b - \gamma_s) & 0 \end{bmatrix} \quad V_{s,S_F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ V_{s,S_H} = \begin{bmatrix} 0 & \sigma(1 - \gamma_b) & 0 \\ 0 & -\sigma^2 \gamma_s(1 - \gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V_{S_F,S_H} = \begin{bmatrix} 0 & 0 & \sigma(\gamma_b - \gamma_s) & 0 \\ 0 & 0 & -\sigma^2(\gamma_b - \gamma_s) & 0 \end{bmatrix} \]

\[ V_{s,u} = \begin{bmatrix} (\varphi + 1)^2 & -(\varphi + 1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \]

By integrating (120) over \([\frac{1}{2}, 1]\)

\[
\frac{1}{2} V_0 = \frac{1 - \theta}{\theta} (1 - \beta \theta) \sum_{t=0}^{\infty} \beta^t E_0 \left[ \int_{\frac{1}{2}}^{1} \hat{s}_i' \hat{d}_i' r_{s,F} + \int_{0}^{\frac{1}{2}} \hat{s}_i' \hat{d}_i' r_{s,H} - \int_{\frac{1}{2}}^{1} \hat{u}_i' \hat{d}_i' r_{s,F} + \int_{0}^{\frac{1}{2}} \hat{s}_i' R_{s,F,s,F} s_i' \hat{d}_i \\
+ \frac{1}{2} \int_{\frac{1}{2}}^{1} \hat{s}_i' R_{s,F,S_F} \int_{\frac{1}{2}}^{1} \hat{s}_i' \hat{d}_i + \frac{1}{2} \int_{0}^{\frac{1}{2}} \hat{s}_i' \hat{d}_i' R_{s,H,S_H} \int_{0}^{\frac{1}{2}} \hat{s}_i' \hat{d}_i + \int_{0}^{\frac{1}{2}} \hat{s}_i' \hat{d}_i' R_{s,F,S_H} \int_{\frac{1}{2}}^{1} \hat{s}_i' \hat{d}_i \\
- \int_{\frac{1}{2}}^{1} \hat{s}_i' R_{s,F,u} \hat{u}_i' \hat{d}_i \right] + s.o.t.i.p. \quad (119)
\]

where:

\[ r_{s,F}' = [\varphi, \sigma \gamma_b, 0] \quad r_{s,H}' = [0, \sigma(1 - \gamma_b), 0] \quad r_{s,u}' = [(\varphi + 1), -1] \]

\[ \begin{align*}
R_{s,F,s,F} &= \begin{bmatrix} \varphi(\varphi + 2) & \sigma \gamma_s & 0 \\ \sigma \gamma_s & -\sigma^2 \gamma_s^2 & 0 \\ 0 & 0 & \varepsilon(\varphi + 1) \end{bmatrix} \\
R_{s,F,S_F} &= \begin{bmatrix} 0 & \sigma(\gamma_b - \gamma_s) & 0 \\ 0 & -\sigma^2(\gamma_b - \gamma_s) & 0 \\ \varepsilon(\varphi + 1) \end{bmatrix} \\
R_{s,H,S_H} &= \begin{bmatrix} 0 & 0 & \sigma(\gamma_b - \gamma_s) \\ 0 & -\sigma^2(\gamma_b - \gamma_s) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
R_{s,F,u} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ (\varphi + 1)^2 & -(\varphi + 1) \end{bmatrix}
\end{align*} \]

Again a symmetric condition can be stated for the regions of the area \( H \) namely:
\[ \frac{1}{2} V_0 = \frac{1 - \theta}{\theta} (1 - \beta \theta) \sum_{t=0}^{\infty} \beta^t E_0 \left[ \int_{\frac{1}{2}}^{1} \hat{s}_i\' \hat{s}_j \kappa_{S_H} + \int_{0}^{\frac{1}{2}} \hat{s}_i\' \hat{s}_j \kappa_{S_F} - \int_{\frac{1}{2}}^{1} \hat{u}_i' \hat{u}_j' \kappa_U + \frac{1}{2} \int_{\frac{1}{2}}^{1} \hat{s}_i' K_{S_H.S_H} s_j' \right. \\
+ \frac{1}{2} \int_{\frac{1}{2}}^{1} \hat{s}_i' K_{S_H.S_H} \int_{\frac{1}{2}}^{1} \hat{s}_j' \, + \frac{1}{2} \int_{0}^{\frac{1}{2}} \hat{s}_i' K_{S_F.S_F} \int_{0}^{\frac{1}{2}} \hat{s}_j' \, + \frac{1}{2} \int_{\frac{1}{2}}^{1} \hat{s}_i' K_{S_H.S_F} \int_{\frac{1}{2}}^{1} \hat{s}_j' \, \\
- \left. \int_{\frac{1}{2}}^{1} \hat{s}_i' K_{S_H,u} \hat{u}_j' \right] + s.o.t.i.p. \]  

(120)

where \( k_{S_H} = r_{S_F}, k_{S_F} = r_{S_H}, k_U = r_U, K_{S_H.S_H} = R_{S_F.S_F}, K_{S_H.S_F} = R_{S_F.S_H}, K_{S_H,U} = R_{S_F.U} \) and \( K_{S_H,U} = R_{S_F.U} \).

Then it can be shown that:

\[ w_s = (1 - \varphi \zeta_b) h_{S_H} - (\xi - \zeta_b) \varphi f_{S_H} - \zeta_b k_{S_H} - (\xi - \zeta_b) r_{S_H} \]

\[ 0 = (1 - \varphi \zeta_b) h_{S_F} - (\xi - \zeta_b) \varphi f_{S_F} - \zeta_b k_{S_F} - (\xi - \zeta_b) r_{S_F} \]  

(121)

where \( \zeta_b = \frac{\frac{1}{2} \varphi + \zeta}{\varphi + \zeta} - \frac{\delta_{3} - 1 + (1/2)^2}{(1 - 2 \theta) \varphi + (1 - 2 \beta) \varphi} \) and \( \xi = \frac{\frac{1}{2} \varphi + \zeta}{\varphi + \zeta} \). Hence we can write the second order approximation of union welfare as:

\[ Y^{\varphi+1} \sum_{t=0}^{\infty} \beta^t E_0 \left[ \frac{1}{2} \int_{\frac{1}{2}}^{1} \hat{s}_i' \Omega_{S_H.S_H} \hat{s}_j' \, + \frac{1}{2} \int_{\frac{1}{2}}^{1} \hat{s}_i' \Omega_{S_F.S_F} \hat{s}_j' \, + \frac{1}{2} \int_{0}^{\frac{1}{2}} \hat{s}_i' \Omega_{S_H.S_H} \hat{s}_j' \, - \int_{\frac{1}{2}}^{1} \hat{s}_i' \Omega_{S_H,u} \hat{u}_j' \right] \]

\[ + t.i.p. \]  

(122)

where

\[ \Omega_{S_H.S_H} \equiv W_{s,s} + (1 - \varphi \zeta_b) H_{S_H.S_H} - (\xi - \zeta_b) \varphi F_{S_H.S_H} - \zeta_b K_{S_H.S_H} \]

\[ \Omega_{S_F.S_F} \equiv (1 - \varphi \zeta_b) H_{S_F.S_F} - (\xi - \zeta_b) \varphi F_{S_F.S_F} - (\xi - \zeta_b) R_{S_F.S_F} \]

\[ \Omega_{S_H.S_F} \equiv (1 - \varphi \zeta_b) H_{S_H.S_F} - (\xi - \zeta_b) \varphi F_{S_H.S_F} - \zeta_b K_{S_H.S_F} - (\xi - \zeta_b) R_{S_H.S_F} \]

\[ \Omega_{S_F.S_H} \equiv (1 - \varphi \zeta_b) H_{S_F.S_H} - (\xi - \zeta_b) \varphi F_{S_F.S_H} - \zeta_b K_{S_F.S_H} - (\xi - \zeta_b) R_{S_F.S_H} \]

\[ \Omega_{S_H,u} \equiv W_{s,u} - \zeta_b K_{S_H,u} \quad \Omega_{S_F,u} \equiv - (\xi - \zeta_b) R_{S_F,u} \]  

(123)

and \( \Omega_{S_H,S_H}, \Omega_{S_F,S_F}, \Omega_{S_H,S_F}, \Omega_{S_F,S_H}, \Omega_{S_H,u} \) and \( \Omega_{S_F,u} \) are respectively equal to:
\[
\begin{bmatrix}
(1 - \zeta_b(\varphi + 1))\varphi & -\zeta_b\gamma_s & 0 \\
-\zeta_b\gamma_s & \omega_sHsH & 0 \\
0 & 0 & \frac{(1 - \zeta_b(\varphi + 1))\epsilon}{\lambda}
\end{bmatrix}
\]
\[
\begin{bmatrix}
-(\xi - \zeta_b)(\varphi + 1)\varphi & -(\xi - \zeta_b)\sigma\gamma_s & 0 \\
-(\xi - \zeta_b)\sigma\gamma_s & \omega_sFsF & 0 \\
0 & 0 & -\frac{((\xi - \zeta_b)(\varphi + 1))\epsilon}{\lambda}
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & -\zeta_b\sigma (\gamma_b - \gamma_s) & 0 \\
-\zeta_b\sigma (\gamma_b - \gamma_s) & \omega_{SHSH} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & -(\xi - \zeta_b)\sigma (\gamma_b - \gamma_s) & 0 \\
-(\xi - \zeta_b)\sigma (\gamma_b - \gamma_s) & \omega_{SF}S & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & -\zeta_b\sigma (1 - \gamma_b) & 0 \\
-(\xi - \zeta_b)\sigma (1 - \gamma_b) & \omega_{SHSF} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
(1 - \zeta_b(\varphi + 1))(\varphi + 1) & \zeta_b(\varphi + 1) \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
-(\xi - \zeta_b)(\varphi + 1)^2 & (\xi - \zeta_b)(\varphi + 1) \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]
with

\[ \omega_{sHsH} \equiv (\sigma - 1) (1 - \tau) \]
\[ + (1 - \zeta_b \varphi) (-\eta \sigma^2 (1 - \gamma_s^2) + \delta_b - \omega_3) \]
\[ - (\xi - \zeta_b) \varphi((1 - \delta_b) + \omega_3) \]
\[ + \zeta_b \sigma^2 \gamma_s^2 \] (124)

\[ \omega_{sFsF} \equiv (1 - \zeta_b \varphi)((1 - \delta_b) + \omega_3) \]
\[ - (\xi - \zeta_b) \varphi(-\eta \sigma^2 (1 - \gamma_s^2) + \delta_b - \omega_3) \]
\[ + (\xi - \zeta_b) \sigma^2 \gamma_s^2 \] (125)

\[ \omega_{SHSH} \equiv (1 - \zeta_b \varphi)(\eta \sigma^2 (1 - \gamma_s^2 - \gamma_b (1 - \gamma_b)) + \omega_3) \]
\[ + (\xi - \zeta_b) \varphi(\eta \sigma^2 \gamma_b (1 - \gamma_b) + \omega_3) \]
\[ + \zeta_b \sigma^2 (\gamma_b^2 - \gamma_s^2) \]
\[ + (\xi - \zeta_b) \sigma^2 (1 - \gamma_b)^2 \] (126)

\[ \omega_{SFSF} \equiv -(1 - \zeta_b \varphi)(\eta \sigma^2 \gamma_b (1 - \gamma_b) + \omega_3) \]
\[ - (\xi - \zeta_b) \varphi(\eta \sigma^2 (1 - \gamma_s^2 - \gamma_b (1 - \gamma_b)) + \omega_3) \]
\[ + \zeta_b \sigma^2 (1 - \gamma_b)^2 \]
\[ + (\xi - \zeta_b) \sigma^2 (\gamma_b^2 - \gamma_s^2) \] (127)

\[ \omega_{SHSF} \equiv (1 - \zeta_b \varphi) \eta \sigma^2 \gamma_b (1 - \gamma_b) \]
\[ - (\xi - \zeta_b) \varphi \eta \sigma^2 \gamma_b (1 - \gamma_b) \]
\[ + \zeta_b \sigma^2 \gamma_b (1 - \gamma_b) \]
\[ + (\xi - \zeta_b) \sigma^2 \gamma_b (1 - \gamma_b) \] (128)

Now we would like to split this welfare approximation in (122) in two components
namely:

\[
Y^{\varphi+1} \sum_{t=0}^{\infty} \beta^t E_0 \left[ \frac{1}{2} \int_0^{1/2} \left( \dot{s}_i^t - \frac{1}{2} \int_{1/2}^{1} \dot{s}_i^t di \right) ' \Omega_{s_H,S_H} \left( \dot{s}_i^t - \frac{1}{2} \int_{1/2}^{1} \dot{s}_i^t di \right) ' \right] + 1/2 \int_{1/2}^{1} \left( \dot{s}_i^t - \frac{1}{2} \int_{1/2}^{1} \dot{s}_i^t di \right) ' \Omega_{s_F,S_F} \left( \dot{s}_i^t - \frac{1}{2} \int_{1/2}^{1} \dot{s}_i^t di \right) ' \] 

\[
- \int_{0}^{1/2} \left( \dot{s}_i^t - \int_{0}^{1/2} \dot{s}_i^t di \right) ' \Omega_{s_H,u} \left( \dot{u}_i^t - \int_{0}^{1/2} \dot{u}_i^t di \right) di 
\]

\[
- \int_{1/2}^{1} \left( \dot{s}_i^t - \int_{1/2}^{1} \dot{s}_i^t di \right) ' \Omega_{s_F,u} \left( \dot{u}_i^t - \int_{1/2}^{1} \dot{u}_i^t di \right) di 
\]

\[
+ \frac{1}{2} \int_{0}^{1/2} \dot{s}_i^t di' \Omega_{s_H,S_H} \int_{0}^{1/2} \dot{s}_i^t di 
\]

\[
- \frac{1}{2} \int_{0}^{1/2} \dot{s}_i^t di' \Omega_{s_H,u} \int_{0}^{1/2} \dot{u}_i^t di 
\]

\[
- \frac{1}{2} \int_{1/2}^{1} \dot{s}_i^t di' \Omega_{s_F,u} \int_{1/2}^{1} \dot{u}_i^t di 
\] 

+ t.i.p. 

(129)

The first component depends only on the average union variables whereas the second depends only on the differences between specific country and average union variables. However this second component can be considered as terms independent of policy (even if they should be taken into account for welfare evaluation) because having as a monetary policy instrument of the average union interest rate, the policy decisions of the Monetary Union Central Bank can just influence the average union economic performance. Thus (129) can be read as:

\[
Y^{\varphi+1} \sum_{t=0}^{\infty} \beta^t E_0 \left[ \frac{1}{2} \int_0^{1/2} \ddot{s}_i^t di' (\Omega_{s_H,S_H} + \Omega_{S_H,S_H}) \int_0^{1/2} \dot{s}_i^t di + \frac{1}{2} \int_{1/2}^{1} \ddot{s}_i^t di' (\Omega_{s_F,S_F} + \Omega_{S_F,S_F}) \int_{1/2}^{1} \dot{s}_i^t di 
\]

\[
+ \int_{0}^{1/2} \ddot{s}_i^t di' \Omega_{s_H,S_F} \int_{0}^{1/2} \dot{s}_i^t di - \int_{0}^{1/2} \ddot{s}_i^t di' \Omega_{s_H,u} \int_{0}^{1/2} \dot{u}_i^t di - \int_{1/2}^{1} \ddot{s}_i^t di' \Omega_{s_F,u} \int_{1/2}^{1} \dot{u}_i^t di 
\] 

+ t.i.p. 

The last step consists in rewriting (130) in terms of gaps with respect to the target of the policymaker of the monetary union. It is easy to show that target is determined by maximizing (130) with respect \( \hat{y}_{H,t}, \hat{y}_{F,t}, \hat{c}_{H,t}, \hat{c}_{F,t} \) and \( \pi_{H,t} \) subject to:

47
\[ \dot{y}_{H,t} = \delta_b \dot{c}_{H,t} + (1 - \delta_b) \dot{c}_{F,t} \quad i \in \left[ 0, \frac{1}{2} \right] \]
\[ \dot{y}_{F,t} = \delta_b \dot{c}_{F,t} + (1 - \delta_b) \dot{c}_{H,t} \quad i \in \left[ \frac{1}{2}, 0 \right] \]  

(130)

In other words, the target of the benevolent central bank of the monetary union coincides with the constrained efficient allocation (namely the allocation that a planner would choose having as objective (130)). According to the first order conditions with respect to \( \dot{y}_{H,t}, \dot{y}_{F,t}, \dot{c}_{H,t}, \dot{c}_{F,t} \) and \( \pi_{H,t} \):

1. \[
(1 - \zeta_b (\varphi + 1)) \varphi \dot{y}_{H,t} - \zeta_b \sigma_1 \left( \gamma_b \dot{c}_{H,t} + (1 - \gamma_b) \dot{c}_{F,t} \right) - (1 - \zeta_b (\varphi + 1)) (\varphi + 1) \dot{a}_{H,t} \\
- \zeta_b (\varphi + 1) \dot{\mu}_{H,t} = \phi_{H,t}^i 
\]

(131)

2. \[
-(\xi - \zeta_b) (\varphi + 1) \dot{y}_{F,t} - (\xi - \zeta_b) \sigma \left( \gamma_b \dot{c}_{H,t} + (1 - \gamma_b) \dot{c}_{F,t} \right) + (\xi - \zeta_b) (\varphi + 1)^2 \dot{a}_{F,t} \\
-(\xi - \zeta_b) (\varphi + 1) \dot{\mu}_{F,t} = \phi_{F,t}^j 
\]

(132)

3. \[
[(\sigma - 1)(1 - \bar{\tau}) + (1 - \zeta_b (\varphi + 1)) \delta_b - (\xi - \zeta_b) \sigma (1 - \delta_b)] \dot{c}_{H,t} - \zeta_b \sigma_1 \gamma_b \dot{y}_{H,t} - (\xi - \zeta_b) \sigma (1 - \gamma_b) \dot{y}_{F,t} \\
+ \zeta_b \sigma^2 \gamma_b \left( \gamma_b \dot{c}_{H,t} + (1 - \gamma_b) \dot{c}_{F,t} \right) + (\xi - \zeta_b) \sigma_1^2 (1 - \gamma_b) \left( \gamma_b \dot{c}_{H,t} + (1 - \gamma_b) \dot{c}_{F,t} \right) \\
(1 - \zeta_b) \eta \sigma_1 \gamma_b (1 - \gamma_b) (\dot{c}_{H,t} - \dot{c}_{F,t}) = -(\delta_b \phi_{H,t}^i + (1 - \delta_b) \phi_{F,t}^i) 
\]

(133)

4. \[
[(1 - \zeta_b (\varphi + 1)) (1 - \delta_b) - (\xi - \zeta_b) \sigma \delta_b \dot{c}_{H,t} - (\xi - \zeta_b) \sigma_1 \gamma_b \dot{y}_{F,t} - \zeta_b \sigma (1 - \gamma_b) \dot{y}_{H,t} \\
+ (\xi - \zeta_b) \sigma_1^2 (1 - \gamma_b) \left( \gamma_b \dot{c}_{H,t} + (1 - \gamma_b) \dot{c}_{F,t} \right) + \zeta_b \sigma_1^2 \gamma_b \left( \gamma_b \dot{c}_{H,t} + (1 - \gamma_b) \dot{c}_{F,t} \right) \\
(1 - \zeta_b) \eta \sigma_1 \gamma_b (1 - \gamma_b) (\dot{c}_{H,t} - \dot{c}_{F,t}) = -(\delta_b \phi_{F,t}^i + (1 - \delta_b) \phi_{H,t}^i) 
\]

(134)

5. \[
(1 - \zeta_b (\varphi + 1)) \frac{\varepsilon}{\lambda} \pi_{H,t} = 0 
\]

(135)

where \( \phi_{H,t}^i \) and \( \phi_{F,t}^j \) are the lagrange multipliers of constraints (47) and (48). Then it can be shown that (122) corresponds to (73) (again by adding and subtracting the target in each term of (122) and then using the conditions just listed) where:

\[ \theta_H \equiv \left[ (\sigma - 1)(1 - \bar{\tau}) + (1 - \zeta_b (\varphi + 1)) \delta_b - (\xi - \zeta_b) \varphi (1 - \delta_b) - (1 - \zeta_b \varphi) \eta \sigma_1^2 \gamma_b (1 - \gamma_b) \\
+ \zeta_b \sigma_1^2 \gamma_b^2 + (\xi - \zeta_b) \sigma_1^2 (1 - \gamma_b)^2 \right] \]
\[ \theta_F \equiv \left[ (1 - \zeta_b (\varphi + 1)) (1 - \delta_b) - (\xi - \zeta_b) \varphi \delta_b + \zeta_b \sigma_1^2 (1 - \gamma_b)^2 + (1 - \zeta_b \varphi) \eta \sigma_1^2 \gamma_b (1 - \gamma_b) \\
+ (\xi - \zeta_b) \sigma_1^2 \gamma_b^2 \right] \]
\[ \theta_{H,F} \equiv (1 - \zeta_b \varphi) \eta \sigma_1^2 \gamma_b (1 - \gamma_b) + \zeta_b \sigma_1^2 \gamma_b (1 - \gamma_b) + (\xi - \zeta_b) \sigma_1^2 (1 - \gamma_b) \gamma_b \]